Behavior of Discount Rates in Present Value Calculation

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Abstract

The aim of this paper is to examine the behavior of the required return (discount rate) in the present value method. This method is regularly used in capital budgeting and investment planning to evaluate the favorability of various decisions. In addition, it is commonly used in the valuation of companies in the form of discounted cash flows. In many cases we notice that the users of this method use deterministic and time-invariant required returns without questioning the assumptions underlying such behavior. We show that the timely behavior of the discount rate depends on the stochastic process that generates the stream of cash flows. We show that required returns (or discount rates) are deterministic and time-invariant if the cash-flow generating process is undisrupted and autoregressive of first order. Other processes generally lead to either time-varying or stochastic discount rates. If the cash flow is generated by a trend-stationary process or specific types of disrupted autoregressive processes, the present value approach can easily be adjusted to produce correct values. More complex cash flows generally require stochastic discount rates. This means that the application of the traditional present value method becomes questionable.

Keywords: Present value method, Required return, Discount rate, Investment appraisal, Mathematical finance

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1 Introduction

In normative present value models found in many standard textbooks on finance, investment analysis or firm valuation (Koller et al., 2010; Hillier at al., 2012; Berk & DeMarzo, 2019; Brealey et al., 2023) it seems to be common that cash flows of limited lifetime are discounted by means of deterministic and constant (time invariant) discount rates. There exists a significant number of empirical articles that deal with time-dependent expected returns. For example, Campbell & Mei (1993) found that the changes in stock prices are more due to changes in expected returns than in cash flows. Timmermann (1995) tested the cointegration of present values and dividends when discount rates are time-varying. Fama & French (2002) and Gagliardini, et al (2016) point at changes of the equity-premium over time; Rosenberg & Engle (2002) estimate time-varying pricing kernels; Tresroci (2013) looks at the time-variation of betas. Based on these empirical observations, it is possible to develop models that could predict present values, discount rates, or both by means of econometric models (Campbell & Schiller, 1988; Geltner & Mei, 1995; Ang & Liu, 2004).

Contrary to this empirical research, we want to explain the variation of required returns (discount rates) purely from the theoretical examination of the behavior of cash flows and continuation values over the lifespan of cash-flow streams. We show that even though pricing kernels, betas, or risk-free rates are constant, discount rates can become time-varying and even stochastic. We want to pinpoint the source of this behavior and give advice how to incorporate this into normative present value models.

Deterministic and constant discount rates in normative present value models require relatively strict assumptions concerning the stochastic properties of streams of cash flows. The first aim of this paper is to show a set of sufficient assumption that allow discount rates to be constant and deterministic. The second aim is to show some circumstances that lead to time-varying or stochastic discount rates. For the user of present value methods, it is important to understand the reasons for both deterministic or stochastic and constant or time varying discount rates, such that it can be considered when calculating present values.

The issues discussed in this paper originally triggered by our research on firm valuation. One of the most used approaches for valuing firms is the concept of discounted cash flow (DCF), which means that present value calculations are at the heart of these methods (Miles & Ezzell, 1980; Inselbag & Kaufold, 1997; Ruback, 2002; Becker, 2022). The researcher or user of DCF methods often examines how present values of equity and debt are affected by changes in the leverage (debt-to-equity financing), which in turn affects interest payments, repayments, tax advantages, costs of financial distress, or the disruption of the cash flows due to bankruptcy. I.e. changing the leverage affects the stochastic behavior of several cash flows that are valued in DCF methods. This quickly leads to complicated situations concerning the estimation of expected cash flows and their corresponding discount rates. In this strain of theoretical research, it has been observed that discount rates can vary over time. Examples are the discount rate of tax shields in Miles & Ezzell (1980), the required return on unlevered and levered equity as well as the weighted average costs of capital in Becker (2022 and 2024).

In this paper we want to take a step back and examine the traditional present value calculation without having to go into the finesses of firm valuation. Particularly, we look at sufficient assumptions that lead to constant required returns. We further analyze how required returns behave if we change these assumptions and provide alternative present value formulas for these cases. We also show when traditional present value calculations are no longer applicable. This basic knowledge is insightful for more complicated problems such as firm valuation.

The sequel of this paper is structured as follows. In the following section we introduce a simple numerical example like in many ordinary textbooks in finance and investment analysis. Afterwards we turn to sufficient assumptions that allow constant discount rates. Particularly, we will see that first-order autoregressive cash flows without disruption are suitable for generating deterministic and constant required returns. Section 4 replaces the autoregressive process with a trend-stationary process. This leads to deterministic but time variant discount rates. Section 5 looks at an autoregressive cash flow that is disrupted. We will see that this case also allows the application of deterministic but time variant discount rates. Here we apply a particular rule for the disruption of the cash-flow stream, but this rule may not seem appealing for real-life applications like firm valuation. Therefore, it is replaced by another rule in section 6. This rule leads to discount rates that are both time-dependent and stochastic. Section 7 concludes this paper.

2 Introductory example & assumptions

We begin our analysis with a simple numerical example. Table 1 presents the expected stream of cash flows that belongs to an investment project. Along with these cash flows, a deterministic (non-stochastic) and constant (time-invariant) discount rate r is given:

Point in time	$t = 0$	$t = 1$	$t = 2$	$t = 3$
Expected cash flow CF_t	n. a.	100	150	200
Discount rate r	10 %	10 %	10 %	n. a.

Table 1: Cash flow and discount rates as input to the present-value calculation

The traditional present value calculation proceeds as follows:

$$\begin{aligned} V_0 &= \frac{CF_1}{1+r} + \frac{CF_2}{(1+r)^2} + \frac{CF_3}{(1+r)^3} \\ &= \frac{100}{1+10\,\%} + \frac{150}{(1+10\,\%)^2} + \frac{200}{(1+10\,\%)^3} \\ &= 365.14 \end{aligned} \tag{1}$$

where V_0 denotes the present value at point in time $t = 0$ of the expected cash flows CF_t at $t = 1, \ldots, 3$. This is the standard approach in many textbooks in finance and investment analysis, whenever we assume discrete points in time (Hillier at al. 2012; Berk & DeMarzo, 2019; Brealey et al., 2023). This approach is easily extended to annuities, perpetuities, or continuous cash flows.

When writing down present-value calculations like (1), two essential aspects are hidden. First, discounting a multi-period stream of cash flows means that we discount two different quantities in each point in time

$t = 1$ to $t = T - 1$, specifically the cash flow at t and the continuation value of all cash flows after t. This becomes apparent, if we use the following equivalent backward iteration for calculating present values:

$$V_2 = \frac{CF_3}{1+r} = \frac{200}{1+10\,\%} = 181.82$$

$$V_1 = \frac{CF_2 + V_2}{1+r} = \frac{150 + 181.82}{1+10\,\%} = 301.65 \tag{2}$$

$$V_0 = \frac{CF_1 + V_1}{1+r} = \frac{150 + 301.65}{1+10\,\%} = 365.14$$

In these calculations we see that the same discount rate is applied to the cash flow alone (particularly CF_3) and for the cash flow and discontinuation joint together (for example $CF_2 + V_2$). This raises the question, which assumptions underly the following approach, which is explicitly used in the backward iteration (2) and implicit in the summation formula (1):

$$V_t = \frac{CF_{t+1} + V_{t+1}}{1 + r_{\mathrm{CV}}} = \frac{CF_{t+1}}{1 + r_{\mathrm{C}}} + \frac{V_{t+1}}{1 + r_{\mathrm{V}}} \quad \text{with} \quad r_{\mathrm{CV}} = r_{\mathrm{C}} = r_{\mathrm{V}}$$

The second aspect that is hidden away, is the stochastic process that generates the expected cash flows in expressions (1) or (2). In Figure 1, we illustrate the stochastic nature behind the expected cash flow (indicated in grey) by means of a scenario tree. This scenario tree considers four points in time, starting at $t = 0$ and ending at $t = 3$. Cash flows appear at points in time $t = 1$ to $t = 3$. We are interested in calculating their value at point in time $t = 0$. The nodes in this tree represent possible states of the world. Departing from some initial node $n = 0$, there is a transition to three possible states $n = 1$ to $n = 3$, and then from each state (node) there will be another transition into one of three possible states, and so forth. The probabilities for transitioning from a parent node to some child node n is denoted by p_n. In addition, a risk neutral probability p_n^* is considered, which we will use to define a pricing kernel. Whenever we refer to some cash flow or value in a particular node of the scenario tree, we use brackets in the subscript, for example $CF_{[2]}$ and $V_{[2]}$ refer to the cash flow and value that appear in node 2. If brackets are absent, then cash flows or values are related to points in time, for example CF_2 and V_2 refer to the cash flow and value that appears at point in time $t = 2$.

In the tree of Figure 1 it becomes transparent that both the cash flow CF_t and continuation value V_t can be stochastic. For example, if we are in a state of the world indicated by node $n = 1$ then we must discount the expectation of the cash flows $CF_{[4]}, CF_{[5]}$, and $CF_{[6]}$. Furthermore, we need to discount the expectation of the continuation values $V_{[4]}, V_{[5]}$, and $V_{[6]}$. It would therefore be legitimate to ask whether the same discount factor can be used for both the expected cash flow and the expected continuation value. If we would reject that $r_{\mathrm{CV}} = r_{\mathrm{C}} = r_{\mathrm{V}}$, then r_{CV} will generally be time-varying even if r_{C} and r_{V} remain constant over time, because the magnitudes of CF_t and V_t relative to each other will generally change over time.

Another legitimate question that becomes apparent from the scenario tree is whether discount rates can be assumed to be deterministic. For example, it cannot be ruled out that the discount rates $r_{\mathrm{C},[1]}$, $r_{\mathrm{C},[2]}$, and $r_{\mathrm{C},[3]}$ (and equivalently $r_{\mathrm{V},[n]}$ and $r_{\mathrm{CV},[n]}$) are all different. In the end they depend on the stochasticity of the cash flows and continuation values in the succeeding point in time.

In other words, if we want to defend the traditional present value formulas like (1) or (2) we need to ask, what are practically reasonable assumptions for this. If these assumptions are not met, can we simply adjust the present value method to give appropriate values? When do traditional present value formulas like (1) and (2) become problematic?

To answer these questions, we introduce a more precise notation for the present value calculations. Particularly, we formulate the following backward iteration process:

$$V_t \mid \mathcal{F}_t = \frac{\mathbb{E}\left[\widetilde{CF}_{t+1} \mid \mathcal{F}_t\right]}{1 + r_{\mathrm{C},t} \mid \mathcal{F}_t} + \frac{\mathbb{E}[\tilde{V}_{t+1} \mid \mathcal{F}_t]}{1 + r_{\mathrm{V},t} \mid \mathcal{F}_t} = \frac{\mathbb{E}\left[\widetilde{CF}_{t+1} \mid \mathcal{F}_t\right] + \mathbb{E}[\tilde{V}_{t+1} \mid \mathcal{F}_t]}{1 + r_{\mathrm{CV},t} \mid \mathcal{F}_t} \tag{3}$$

where $\mathcal{F}_t$ describes the information that is available at point in time t, at which we calculate the expectation $\mathbb{E}$ of the stochastic cash flow $\widetilde{CF}_{t+1}$ or the stochastic continuation value $\tilde{V}_{t+1}$. All discount rates also depend on the state of the world (the information available) at t.

A special case appears if discount rates are deterministic. We can then simply replace $\mathcal{F}_t$ with $\mathcal{F}_0$ on the discount rates. This yields:

$$V_t \mid \mathcal{F}_t = \frac{\mathbb{E}\left[\widetilde{CF}_{t+1} \mid \mathcal{F}_t\right]}{1 + r_{\mathrm{C},t} \mid \mathcal{F}_0} + \frac{\mathbb{E}[\tilde{V}_{t+1} \mid \mathcal{F}_t]}{1 + r_{\mathrm{V},t} \mid \mathcal{F}_0} = \frac{\mathbb{E}\left[\widetilde{CF}_{t+1} \mid \mathcal{F}_t\right] + \mathbb{E}[\tilde{V}_{t+1} \mid \mathcal{F}_t]}{1 + r_{\mathrm{CV},t} \mid \mathcal{F}_0} \tag{4}$$

A further simplification appears if both the conditional expectations of cash flows and continuation values become deterministic, i.e. everything depends only on $\mathcal{F}_0$, and we can basically omit the $\mathcal{F}$-notation:

$$V_t = \frac{\mathbb{E}\left[\widetilde{CF}_{t+1}\right]}{1 + r_{\mathrm{C},t}} + \frac{\mathbb{E}[\tilde{V}_{t+1}]}{1 + r_{\mathrm{V},t}} = \frac{\mathbb{E}\left[\widetilde{CF}_{t+1}\right] + \mathbb{E}[\tilde{V}_{t+1}]}{1 + r_{\mathrm{CV},t}} \tag{5}$$

This can be equivalently expressed by means of the following summation formula:

$$V_0 = \frac{\mathbb{E}\left[\widetilde{CF}_1\right]}{1 + r_{\mathrm{CV},0}} + \frac{\mathbb{E}\left[\widetilde{CF}_2\right]}{\left(1 + r_{\mathrm{CV},0}\right) \cdot \left(1 + r_{\mathrm{CV},1}\right)} + \cdots + \frac{\mathbb{E}\left[\widetilde{CF}_T\right]}{\left(1 + r_{\mathrm{CV},0}\right) \cdot \left(1 + r_{\mathrm{CV},1}\right) \cdot \ldots \cdot \left(1 + r_{\mathrm{CV},T-1}\right)} \tag{6}$$

Finally, the simplest case appears with $r_{\mathrm{CV},0} = r_{\mathrm{CV},1} = \cdots = r_{\mathrm{CV},T-1}$ which brings us back to expression (1), with a slightly more precise notation that highlights the expectations:

$$V_0 = \frac{\mathbb{E}\left[\widetilde{CF}_1\right]}{1 + r_{\mathrm{CV}}} + \frac{\mathbb{E}\left[\widetilde{CF}_2\right]}{(1 + r_{\mathrm{CV}})^2} + \cdots + \frac{\mathbb{E}\left[\widetilde{CF}_T\right]}{(1 + r_{\mathrm{CV}})^T} \tag{7}$$

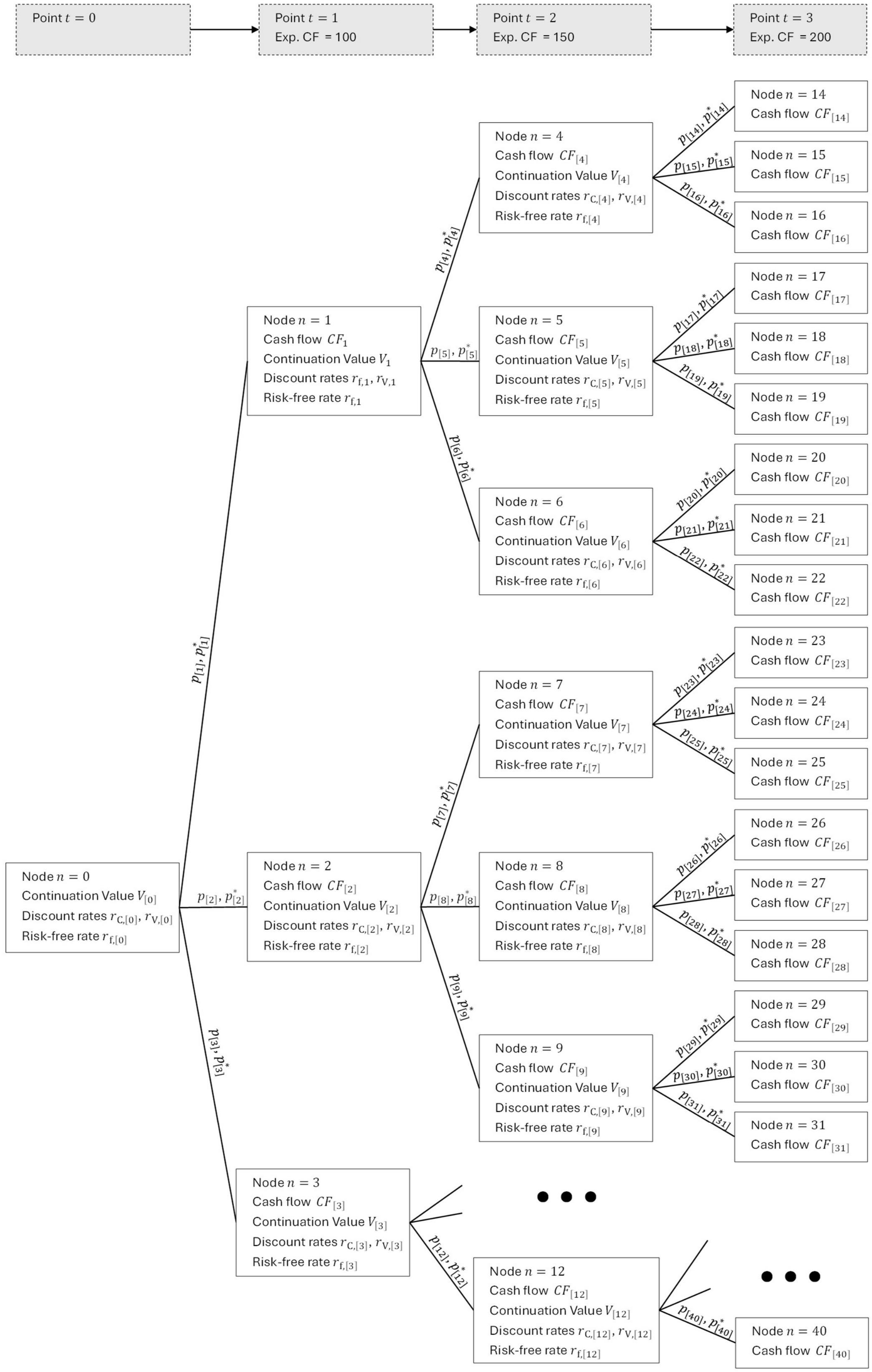

Figure 1: Representation of the expected cash-flow stream by means of a scenario tree

Before, we state reasonable assumptions, we shortly look into the definition of the discount rates, for which we make use of the first fundamental theorem of asset pricing (Pascucci, 2011). It states that a market is free of arbitrage if there exists at least one risk neutral probability measure that is equivalent to the original probability measure. This means that value V_t in expression (3), can be equally computed by the following expression:

$$V_t \mid \mathcal{F}_t = \frac{\mathbb{E}^*\left[\widetilde{CF}_{t+1} \mid \mathcal{F}_t\right]}{1 + r_{\mathrm{f},t} \mid \mathcal{F}_t} + \frac{\mathbb{E}^*[\tilde{V}_{t+1} \mid \mathcal{F}_t]}{1 + r_{\mathrm{f},t} \mid \mathcal{F}_t} = \frac{\mathbb{E}^*\left[\widetilde{CF}_{t+1} \mid \mathcal{F}_t\right] + \mathbb{E}^*[\tilde{V}_{t+1} \mid \mathcal{F}_t]}{1 + r_{\mathrm{f},t} \mid \mathcal{F}_t} \tag{8}$$

where $\mathbb{E}^*$ is the risk-neutral conditional expectation and $r_{\mathrm{f},t}$ is the risk-free rate. Let us refer to the probability measures underlying $\mathbb{E}$ and $\mathbb{E}^*$ as $\mathcal{P}$ (original probability measure) and $\mathcal{P}^*$ (risk neutral probability measure). Looking at the tree in Figure 1, the probability measure $\mathcal{P}_{[1]}$ consists of the probabilities $p_{[4]}$, $p_{[5]}$ and $p_{[6]}$. The risk-neutral measure $\mathcal{P}^*_{[1]}$ consists of the probabilities $p^*_{[4]}$, $p^*_{[5]}$ and $p^*_{[6]}$.

In what follows, we make two assumptions that are maintained throughout the remainder of this paper.

Assumption 1 – Additivity of present values: We assume that present values are additive:

$$V\left[\widetilde{CF}_A + \widetilde{CF}_B\right] = V\left[\widetilde{CF}_A\right] + V\left[\widetilde{CF}_B\right]$$

The additivity of present values is based on the arbitrage-free nature of capital markets (Varian, 2012). A special case of this additivity appears if $\widetilde{CF}_B = (c-1) \cdot \widetilde{CF}_A$ where $c-1$ is a constant. With this, we want to say that $\widetilde{CF}_B$ is a multiple of $\widetilde{CF}_A$, and we can state that:

$$V\left[c \cdot \widetilde{CF}_A\right] = c \cdot V\left[\widetilde{CF}_A\right]$$

Assumption 2a – Deterministic probability measures and risk-free rate (constant pricing kernel throughout time). Particularly, we assume that the distribution measures underlying the expectations in (3) and (8) as well as the risk-free rate are deterministic, i. e.

$$(\mathcal{P}_t \mid \mathcal{F}_t) = (\mathcal{P}_t \mid \mathcal{F}_0), \qquad (\mathcal{P}^*_t \mid \mathcal{F}_t) = (\mathcal{P}^*_t \mid \mathcal{F}_0), \qquad \left(r_{\mathrm{f},t} \mid \mathcal{F}_t\right) = \left(r_{\mathrm{f},t} \mid \mathcal{F}_0\right)$$

For the tree in Figure 1 this means that all the probability measures of a particular point in time are the same. For example, at $t = 1$, we would have $\mathcal{P}_{[1]} = \mathcal{P}_{[2]} = \mathcal{P}_{[3]}$ and $\mathcal{P}^*_{[1]} = \mathcal{P}^*_{[2]} = \mathcal{P}^*_{[3]}$. In addition, we assume that $r_{\mathrm{f},[1]} = r_{\mathrm{f},[2]} = r_{\mathrm{f},[3]}$.

There exists empirical evidence that pricing kernels do not remain constant over time (Rosenbaum and Engle, 2002). I.e. pricing kernels do not only vary in time but are unforeseeable or unpredictable. Clearly, pricing kernels represent aggregated preferences and beliefs concerning return and risk, and will be subject to changes. However, a straightforward use of the traditional present value analysis like formula (1) is problematic if we assume stochastic pricing kernels, since we would have to deal with stochastic discount rates, i.e.

$$\mathbb{E}\left[\tilde{V}_t\left(\widetilde{CF}_{t+1} \mid \mathcal{F}_t\right) \mid \mathcal{F}_{t-1}\right] = \mathbb{E}\left[\frac{\mathbb{E}\left[\widetilde{CF}_{t+1} \mid \mathcal{F}_t\right]}{1 + \tilde{r}_t \mid \mathcal{F}_t} \mid \mathcal{F}_{t-1}\right] \neq \frac{\mathbb{E}\left[\widetilde{CF}_{t+1} \mid \mathcal{F}_{t-1}\right]}{\mathbb{E}[1 + \tilde{r}_t \mid \mathcal{F}_{t-1}]} \tag{9}$$

In other words, we cannot discount expected cash flows with expected required returns, unless required returns are deterministic. Of course, we can find a discount rate r' that satisfies the following equation:

$$\mathbb{E}\left[\tilde{V}_t\left(\widetilde{CF}_{t+1} \mid \mathcal{F}_t\right) \mid \mathcal{F}_{t-1}\right] = \mathbb{E}\left[\frac{\mathbb{E}\left[\widetilde{CF}_{t+1} \mid \mathcal{F}_t\right]}{1+\tilde{r}_t \mid \mathcal{F}_t} \mid \mathcal{F}_{t-1}\right] = \frac{\mathbb{E}\left[\widetilde{CF}_{t+1} \mid \mathcal{F}_{t-1}\right]}{1+r'}$$

However, the question remains how to interpret r' because $\mathbb{E}[1+\tilde{r}_t \mid \mathcal{F}_{t-1}]$ represents the expected cost of capital (opportunity costs), but r' does not have this convenient economic meaning.

Anyway, the purpose of this paper is not to discuss the treatment of stochastic pricing kernels in traditional present value calculations, but to study whether discount rates can turn out to be stochastic even though the pricing kernel is deterministic.

Assumption 2b – Deterministic probability measures and risk-free rate (constant pricing kernel throughout time). Particularly, we assume that the distribution measures underlying the expectations in (3) and (8) as well as the risk-free rate are constant throughout time, i. e.

$$\mathcal{P}_t \mid \mathcal{F}_0 = \mathcal{P}\,, \qquad \mathcal{P}_t^* \mid \mathcal{F}_0 = \mathcal{P}^*\,, \qquad r_{\mathrm{f},t} \mid \mathcal{F}_0 = r_{\mathrm{f}} \quad \text{for all } t$$

This assumption is not as critical as assumption 2a. It can be easily removed. Indeed, valuation practitioners may be tempted to consider the term structure of interest rates in their valuation models. However, the purpose of the paper is to show that discount rates will vary even though the pricing kernel is constant. In other words, we want to show that variations in discount rates must not solely be prescribed to changes in the pricing kernel.

To build a valid present value model, we must add some more assumptions related to the stochastic nature of cash flows. In the following sections, we present different such assumptions and study their effect on the behavior of discount rates in present value calculations.

3 Undisrupted autoregressive cash flows

In this section we add the following assumption:

Assumption 3a – undisrupted first-order autoregressive stream of cash flows: The cash flow follows an autoregressive process of first order of the following form:

$$\tilde{z}_t = z_{t-1} \cdot \tilde{\varepsilon}_t$$

$$\widetilde{CF}_t = a_t \cdot \tilde{z}_t$$

where $\tilde{\varepsilon}_t$ is a stochastic input-parameter that is drawn independently from the same time-invariant distribution $\mathcal{D}$ (this means $\tilde{\varepsilon}_t \sim \mathcal{D}$ for all t). The variable $\tilde{z}_t$ can be interpreted as a basis cash flow, and a_t is an additional deterministic scaling parameter, that allows cash flows to grow or shrink without changing its underlying stochasticity. It is important to note that these equations are stated as if we are at point in time $t-1$ where we already have observed the precise realization of z_{t-1}. From the perspective of $t-2$, the precise value of $\tilde{z}_{t-1}$ has not yet been observed. We assume that some initial value z_0 is known.

We write the autoregressive property of the cash flows more directly as follows:

$$\widetilde{CF}_t = CF_{t-1} \cdot \tilde{\varepsilon}_t \cdot b_t \qquad \text{where} \qquad b_t = \frac{a_t}{a_{t-1}}$$

It is worth to mention, that $\tilde{\varepsilon}_t = 0$ not only forces $CF_t = 0$, but it sets all future cash flows from $t+1$ and beyond to zero. This can be interpreted as a disruption of the cash flow. Such a situation is shown in Figure 3 which will be discussed below. Apart from this special case, the cash flow is never disrupted and continues until infinity or some specified end of the lifetime of the cash-flow stream.

In what follows we analyze the effects of assumptions 1, 2, and 3a on the discount rate. We start our evaluation process in the last point in time. Here we must determine the value of the cash flow $\widetilde{CF}_T$ which according to assumption 3a is defined as: $\widetilde{CF}_T = CF_{T-1} \cdot \tilde{\varepsilon}_T \cdot b_T$. The value is determined as follows:

$$V_{T-1} = \frac{\mathbb{E}_{T-1}\left[\widetilde{CF}_T\right]}{1+r_\mathrm{C}} = \frac{CF_{T-1} \cdot b_T \cdot \mathbb{E}_{T-1}[\tilde{\varepsilon}_T]}{1+r_\mathrm{C}}$$

The subscript $T-1$ on the expectation operator $\mathbb{E}$ refers to the point in time at which we calculate this expectation. In other words, $\mathbb{E}_{T-1}\left[\widetilde{CF}_T\right]$ is the conditional expectation that can be calculated using the information $\mathcal{F}_{T-1}$ available at $T-1$. The discount rate r_C represents the required return that is applied to any stochastic variable that is proportional to $\tilde{\varepsilon}_T$. This is where we require assumption 1:

$$V_{T-1}[\tilde{\varepsilon}_T] = \frac{\mathbb{E}_{T-1}[\tilde{\varepsilon}_T]}{1+r_\mathrm{C}} \qquad \rightarrow \qquad V_{T-1}\left[\widetilde{CF}_T\right] = V_{T-1}[CF_{T-1} \cdot b_T \cdot \tilde{\varepsilon}_T]$$

$$= CF_{T-1} \cdot b_T \cdot V_{T-1}[\tilde{\varepsilon}_T]$$

$$= \frac{CF_{T-1} \cdot b_T \cdot \mathbb{E}_{T-1}[\tilde{\varepsilon}_T]}{1+r_\mathrm{C}}$$

Now we move backwards to point in time $T-2$ where we can calculate the value of all future cash flows as follows:

$$V_{T-2}=\frac{\mathbb{E}_{T-2}\left[\widetilde{CF}_{T-1}\right]+\mathbb{E}_{T-2}\left[\tilde{V}_{T-1}\right]}{1+r_{\mathrm{CV},T-2}}=\frac{\mathbb{E}_{T-2}\left[\widetilde{CF}_{T-1}\right]}{1+r_{\mathrm{C}}}+\frac{\mathbb{E}_{T-2}\left[\widetilde{CF}_{T-1}\cdot b_T\cdot\mathbb{E}_{T-1}[\tilde{\varepsilon}_T]\right]}{(1+r_{\mathrm{C}})\cdot\left(1+r_{\mathrm{V},T-2}\right)}$$

where the values of $r_{\mathrm{CV},T-2}$ and $r_{\mathrm{V},T-2}$ are preliminarily unknown and need to be established. Because $\tilde{\varepsilon}_T$ is independently and identically distributed, its expectation seen from $T-2$ is the same as seen from $T-1$. This means, as we move from $T-2$ to $T-1$, the expectation $\mathbb{E}_{T-1}[\tilde{\varepsilon}_T]$ does not change; it is deterministic. We make this clearer by denoting $\mathbb{E}_{T-2}\left[\mathbb{E}_{T-1}[\tilde{\varepsilon}_T]\right]=\mathbb{E}_{T-2}[\tilde{\varepsilon}_T]=\bar{\varepsilon}$. We therefore write:

$$V_{T-2}=\frac{\mathbb{E}_{T-2}\left[\widetilde{CF}_{T-1}\right]}{1+r_{\mathrm{C}}}+\frac{b_T\cdot\bar{\varepsilon}\cdot\mathbb{E}_{T-2}\left[\widetilde{CF}_{T-1}\right]}{(1+r_{\mathrm{C}})^2}$$

Where $\widetilde{CF}_{T-1}$ is the only stochastic variable that is resolved during the transition from $T-2$ to $T-1$. Therefore, both the expected cash flow $\mathbb{E}_{T-2}\left[\widetilde{CF}_{T-1}\right]$ and the expected continuation value $\mathbb{E}_{T-2}\left[\tilde{V}_{T-1}\right]$ need to be discounted with the same required return, i. e. $r_{\mathrm{C}}=r_{\mathrm{V}}=r_{\mathrm{CV}}$. Here, we have used assumption 2 concerning the constant pricing kernel. This implies that r_{CV} remains constant over time.

Figure 2 illustrates how the cash flows and continuation values evolve throughout time. The stochastic development is driven by the following input parameters: The initial basis cash flow is $z_0=100$; The stochastic term is $\tilde{\varepsilon}=\{2.3;0.8;\ -0.1\}$, where each value can appear with the same probability of 1/3. The scaler takes the values: $a_1=1$, $a_2=1.5$, and $a_3=2$. This gives the unconditional expected cash flows as shown in Table 1.

Let us verify some of the calculations. The cash flows in nodes 13 to 15 are determined as follows (We use brackets to indicate that we refer to nodes in the scenario tree instead of time periods):

$$CF_{[13]}=z_0\cdot\varepsilon_{[1]}\cdot\varepsilon_{[4]}\cdot\varepsilon_{[13]}\cdot a_2=100\cdot 2.3\cdot 2.3\cdot 2.3\cdot 2=2{,}433.40$$

$$CF_{[14]}=z_0\cdot\varepsilon_{[1]}\cdot\varepsilon_{[4]}\cdot\varepsilon_{[14]}\cdot a_2=100\cdot 2.3\cdot 2.3\cdot 0.8\cdot 2=846.40$$

$$CF_{[15]}=z_0\cdot\varepsilon_{[1]}\cdot\varepsilon_{[4]}\cdot\varepsilon_{[15]}\cdot a_2=100\cdot 2.3\cdot 2.3\cdot(-0.1)\cdot 2=-105.80$$

The expected cash flow at $t=3$ conditional on the information in $t=2$ is therefore:

$$\mathbb{E}_2[CF_3]=\frac{1}{3}\cdot 2{,}433.40+\frac{1}{3}\cdot 846.40+\frac{1}{3}\cdot(-105.80)=1{,}058.00$$

Note that the unconditional expected cash flow for $t=3$ is $\mathbb{E}_0[CF_3]=200$ (the average of the cash flows in all nodes from 13 to 39).

The continuation value in node 4 at $t=2$ of the expected cash flow at $t=3$ is:

$$V_{[4]}=\frac{1{,}058}{1+10\,\%}=961.82$$

Accordingly, the values in nodes 5 and 6 are calculated. The conditional expectation in node 1 at $t=1$ of these values is as follows:

$$\mathbb{E}_{[1]}[\tilde{V}_2]=\frac{1}{3}\cdot 961.82+\frac{1}{3}\cdot 334.55+\frac{1}{3}\cdot(-41.82)=418.18$$

The notation $\mathbb{E}_{[1]}[\tilde{V}_2]$ means that we are defining the conditional expectation in node 1 of the potentially stochastic value in period $t = 2$. Let us now turn to the expected cash flow at $t = 2$ conditional on node 1 is:

$$\mathbb{E}_{[1]}\left[\widetilde{CF}_2\right] = \frac{1}{3} \cdot 793.50 + \frac{1}{3} \cdot 276.00 + \frac{1}{3} \cdot (-34.50) = 345.00$$

The value in node 1 is then based on the following calculation:

$$V_{[1]} = \frac{418.18 + 345.00}{1 + 10\,\%} = 693.80$$

Accordingly, the values in nodes 2 and 3 can be calculated. Finally, the value in node 0 at $t = 0$ becomes:

$$V_{[0]} = \frac{\frac{1}{3} \cdot (230 + 693.80) + \frac{1}{3} \cdot (80 + 241.32) + \frac{1}{3} \cdot (-10 - 30.17)}{1 + 10\,\%} = 365.14$$

As we see, we could have applied the traditional present value formula shown in (1). It gives the same result. We can conclude that assumption 1, 2, and 3a allow the computation of present values with a constant (time invariant) and deterministic discount rate.

Certainly, Assumptions 1, 2, and 3a are only sufficient, but not necessary for constant discount rates. However, they represent a somewhat stable and effective environment for valuation approaches. Of course, we can imagine that time-variant or stochastic pricing kernel coincides with some stochastic cash flow in such a way that the discount rates become constant. However, if we evaluate several streams of cash flows like in the analysis of mutually exclusive investment projects, we may have problems arguing why all streams of cash flows coincidently have constants discount rates.

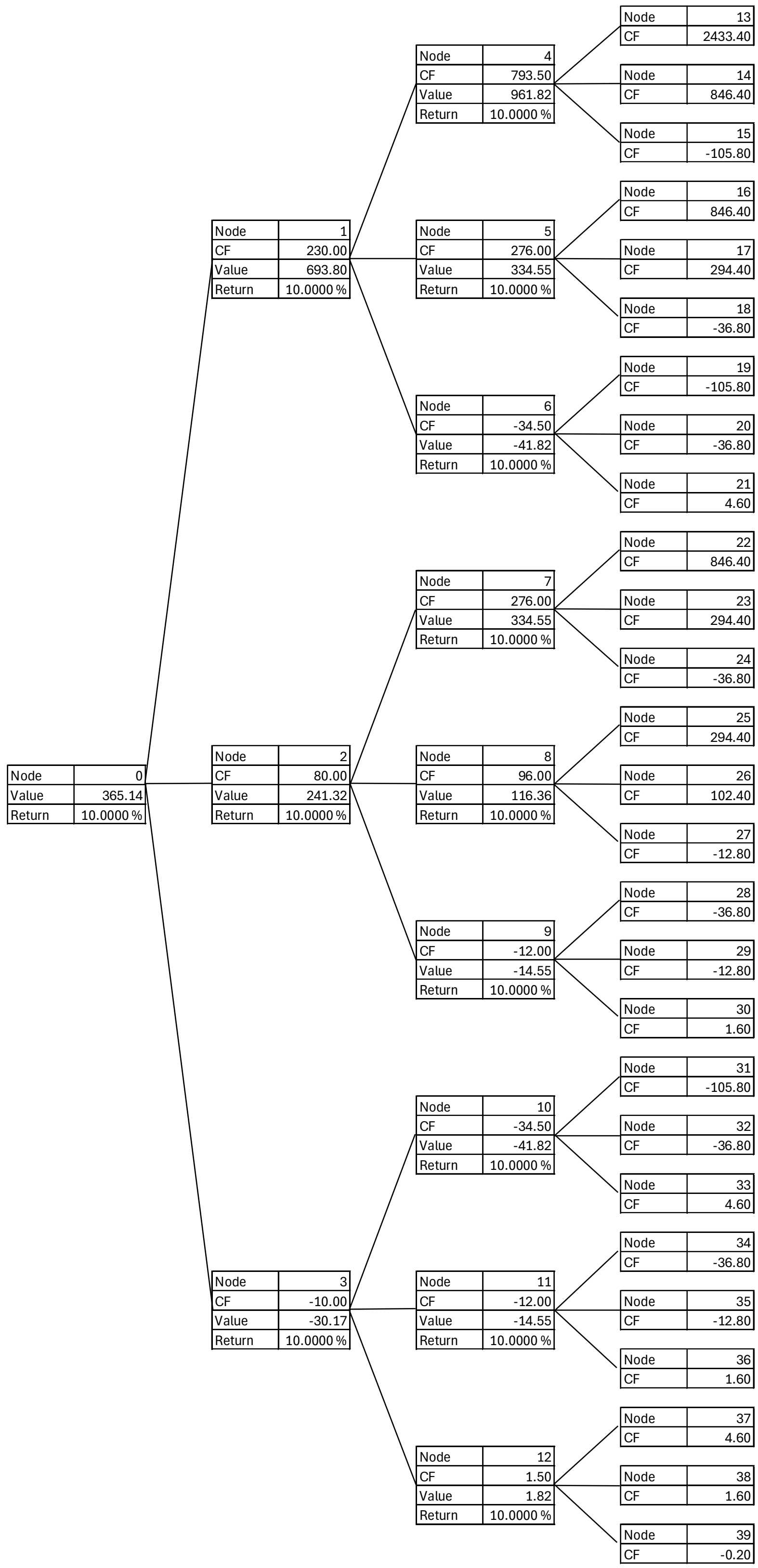

Figure 2: Evolution of cash flows, values, and discount rates in the non-disrupted auto-regressive case

Before, we move to another stochastic process, we want to look at the special case where $\varepsilon_t = 0$ with some positive probability. This situation is shown in the scenario tree in Figure 3. In nodes, 3, 6 and 9 we see that the future stream of cash flows becomes zero because $\varepsilon_t = 0$. This essentially means that the future cash flow is discontinued, and the continuation values in nodes 3, 6 and 9 are zero. Normally, a discount rate (return) is not defined for situations where both the expected cash flow and continuation value become Zero with certainty:

$$0 = \frac{0 + 0}{1 + r}$$

We may even tend to discount a certain cash flow of zero by the risk-free rate. However, since we can discount a cash flow (or continuation value) of zero with any discount rate, and still obtain a value of zero, we can apply the present value formula with a constant required return. Note that the constant required return in Figure 3 is different from 10 %, particularly it is $r_{\mathrm{C}} = 9.6$ %. This is because we use another stochastic term, particularly $\tilde{\varepsilon} = \{2.3; 0.8;\ 0\}$. The reasoning behind this discount rate is shown in Appendix B.

A critical aspect of first-order autoregressive processes with possible $\varepsilon_t = 0$ is that future cash flows can never become different from zero after point in time t once they have become zero in t, and such a behavior might not be desired. In section 4, we will therefore look at another simple stochastic process.

Another critical issue is the following: Once the cash flow becomes negative there is a high probability that negative cash flows will follow. In practice investment projects can be stopped, if future prospects look poor. We therefore look at disrupted cash flows in sections 5 and 6.

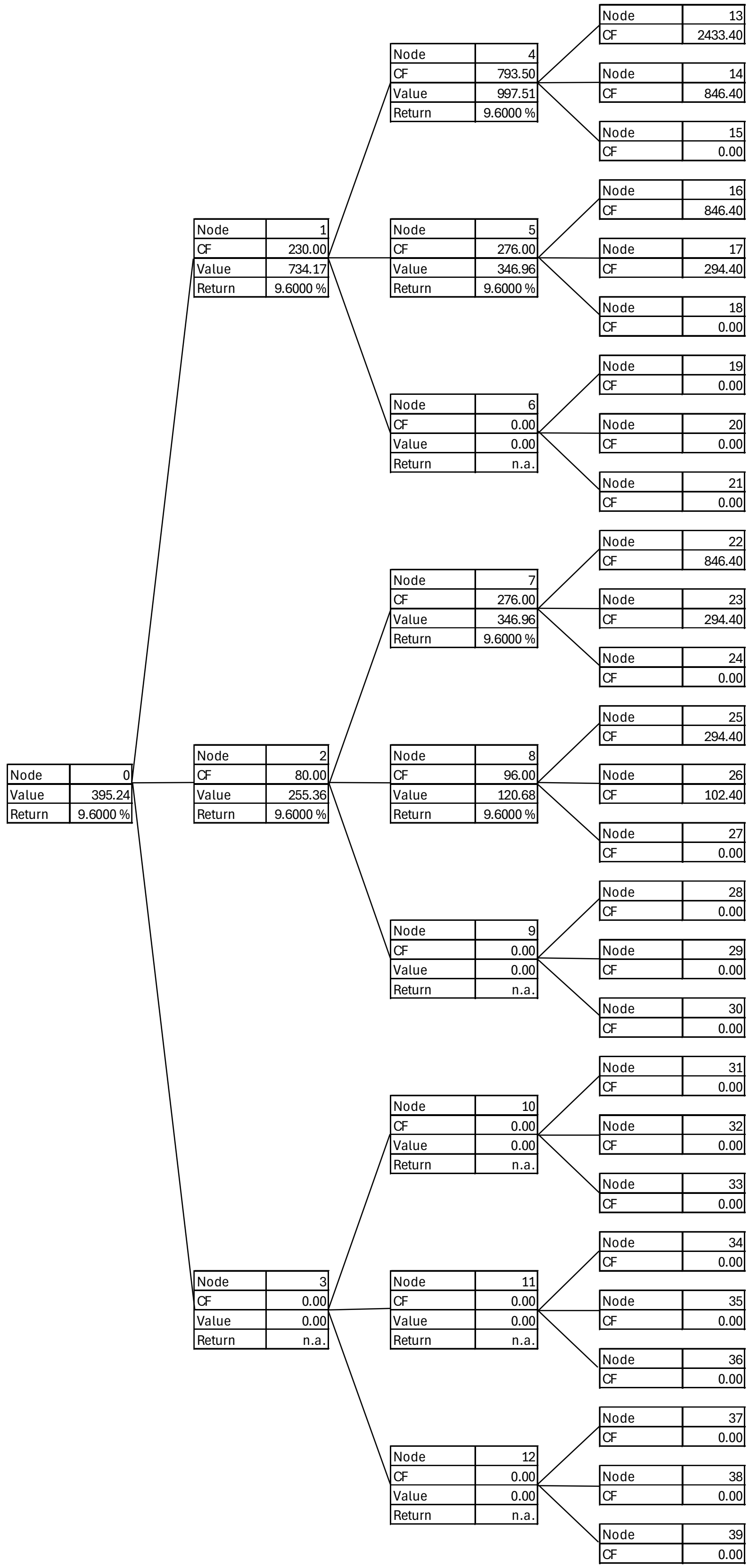

Figure 3: Disruption of the cash-flow stream in case of $\varepsilon = 0$

4 Undisrupted trend-stationary cash flows

Contrary to the autoregressive property of the cash flow according to assumption 3a, we now assume that the cash flow behaves according to the following stochastic process:

Assumption 3b – undisrupted trend-stationary stream of cash flows: The cash flow follows a trend-stationary process of the following form:

$$\widetilde{CF}_t = \tilde{\varepsilon}_t \cdot a_t \quad \text{with} \quad \mathbb{E}_{t-1}[\tilde{\varepsilon}_t] = 1 \tag{10}$$

where $\mathbb{E}_{t-1}$ denotes the expectation observable at point in time $t-1$ of the random term $\tilde{\varepsilon}_t$ that is drawn independently from the same time-invariant distribution $\mathcal{D}$ (i.e. $\tilde{\varepsilon}_t \sim \mathcal{D}$ for all t). The deterministic parameter a_t scales the cash flow up or down. This process implies that neither the cash flow $\widetilde{CF}_t$ nor $\tilde{\varepsilon}_t$ observed at t depend on information at preceding points in time. We refer to this process as a trend-stationary process, since a scaler $a_t = 1$ would leave us with a strictly stationary stream of cash flows.

Let us now find the value of this stream of cash flows. We begin with the calculation of the value at $t = T-1$ of the stochastic cash flow at T:

$$V_{T-1} = \frac{\mathbb{E}_{T-1}\left[\widetilde{CF}_T\right]}{1+r_{\mathrm{C}}} = \frac{a_T \cdot \mathbb{E}_{T-1}[\tilde{\varepsilon}_T]}{1+r_{\mathrm{C}}}$$

Like in the previous section we discount the expectation of the stochastic variable $\tilde{\varepsilon}_t$ with the discount rate r_{C}. Now we move backwards in time where we have:

$$V_{T-2} = \frac{\mathbb{E}_{T-2}\left[\widetilde{CF}_{T-1}\right] + \mathbb{E}_{T-2}[\tilde{V}_{T-1}]}{1+r_{\mathrm{CV},T-2}} = \frac{\mathbb{E}_{T-2}\left[\widetilde{CF}_{T-1}\right]}{1+r_{\mathrm{C}}} + \frac{\mathbb{E}_{T-2}\left[a_T \cdot \mathbb{E}_{T-1}[\tilde{\varepsilon}_T]\right]/(1+r_{\mathrm{C}})}{\left(1+r_{\mathrm{V},T-2}\right)}$$

where $r_{\mathrm{CV},T-2}$ and $r_{\mathrm{V},T-2}$ are preliminarily unknown discount rates which need to be established. Like in the previous section, $\mathbb{E}_{T-2}\left[\mathbb{E}_{T-1}[\tilde{\varepsilon}_T]\right] = \mathbb{E}_{T-2}[\tilde{\varepsilon}_T] = \bar{\varepsilon}_T$. We see clearly that the continuation value does not depend on any information at preceding points in time; It is deterministic and therefore must be discounted with the risk-free rate. This means, that we have two parts that need to be discounted separately: (1) the cash flow that requires a return r_{C}, and (2) the continuation value that requires the risk-free rate $r_{\mathrm{V},T-2} = r_{\mathrm{f}}$:

$$V_{T-2} = \frac{\mathbb{E}_{T-2}\left[\widetilde{CF}_{T-1}\right] + \mathbb{E}_{T-2}[\tilde{V}_{T-1}]}{1+r_{\mathrm{CV},T-2}} = \frac{\mathbb{E}_{T-2}\left[\widetilde{CF}_{T-1}\right]}{1+r_{\mathrm{C}}} + \frac{\mathbb{E}_{T-2}[\tilde{V}_{T-1}]}{1+r_{\mathrm{f}}}$$

We can continue with backward iteration, and receive the following general equation for the calculation of values:

$$V_{t-1} = \frac{\mathbb{E}_{t-1}\left[\widetilde{CF}_t\right] + V_t}{1+r_{\mathrm{CV},t-1}} = \frac{\mathbb{E}_{t-1}\left[\widetilde{CF}_t\right]}{1+r_{\mathrm{C}}} + \frac{V_t}{1+r_{\mathrm{f}}} \tag{11}$$

We see that the discount rate $r_{\mathrm{CV},t-1}$ in this recursive formulation can change over time, and its magnitude depends on the magnitudes of the expected cash flow $\mathbb{E}\left[\widetilde{CF}_t\right]$ and deterministic continuation value V_t.

Particularly, the discount rate $r_{\mathrm{CV},t-1}$ can be calculated as follows:

$$r_{\mathrm{CV},t-1} = \frac{V_{t-1}}{\mathbb{E}_{t-1}\left[\widetilde{CF}_t\right] + V_t} + 1$$

However, in this case we have already computed V_{t-1}, and do not need the discount rate $r_{\mathrm{CV},t-1}$ anymore. However, we can define the following factors:

$$v_t = \frac{\mathbb{E}_{t-1}\left[\widetilde{CF}_t\right]}{\mathbb{E}_{t-1}\left[\widetilde{CF}_t\right] + \mathbb{E}_{t-1}[\tilde{V}_t]}, \quad 1 - v_t = \frac{\mathbb{E}_{t-1}[\tilde{V}_t]}{\mathbb{E}_{t-1}\left[\widetilde{CF}_t\right] + \mathbb{E}_{t-1}[\tilde{V}_t]} \tag{12}$$

Substituting these into (11) yields:

$$\frac{1}{1 + r_{\mathrm{CV},t-1}} = \frac{v_t}{1 + r_{\mathrm{C}}} + \frac{1 - v_t}{1 + r_{\mathrm{f}}}$$

After solving for $r_{\mathrm{CV},t-1}$ we obtain:

$$r_{\mathrm{CV},t-1} = \frac{(1 - v_t) \cdot r_{\mathrm{f}} + v_t \cdot r_{\mathrm{C}} + r_{\mathrm{C}} \cdot r_{\mathrm{f}}}{1 + v_t \cdot r_{\mathrm{f}} + (1 - v_t) \cdot r_{\mathrm{C}}} \tag{13}$$

This means that we can establish the discount rate $r_{\mathrm{CV},t-1}$ as some kind of weighted average of r_{f} and r_{C}.

Let us now turn to the calculation in Figure 4. This figure illustrates how the cash flows and continuation values evolve throughout time. The stochastic development is driven by the following input parameters: The stochastic term is $\tilde{\varepsilon} = \{230; 80; -10\}$, where each value can appear with the same probability of 1/3. The scaler takes the values: $a_1 = 1$, $a_2 = 1.5$, and $a_3 = 2$. This gives the expected cash flows as shown in Table 1.

The cash flows in node 13 to 15 (and equally 16 to 18, 19 to 21, and so forth) are determined as follows:

$$CF_{[13]} = CF_{[16]} = \cdots CF_{[37]} = 230 \cdot 2 = 460$$

$$CF_{[14]} = CF_{[17]} = \cdots CF_{[38]} = 80 \cdot 2 = 160$$

$$CF_{[15]} = CF_{[18]} = \cdots CF_{[39]} = -10 \cdot 2 = -20$$

The conditional expectation equals the unconditional expectation:

$$\mathbb{E}_2[CF_3] = \frac{1}{3} \cdot 460 + \frac{1}{3} \cdot 160 + \frac{1}{3} \cdot -20 = 200$$

The values in all nodes at $t = 2$ are the same:

$$V_{[4]} = V_{[5]} = \cdots V_{[12]} = \frac{200}{1 + 10\ \%} = 181.82$$

Likewise, we can calculate the value of all nodes in $t = 1$:

$$V_{[1]} = V_{[2]} = V_{[3]} = \frac{150}{1 + 10\ \%} + \frac{181.82}{1 + 5\ \%} = 309.52$$

Continuing this approach, we will get to node 0 at $t = 0$, where we have:

$$V_{[0]} = \frac{100}{1 + 10\ \%} + \frac{309.52}{1 + 5\ \%} = 385.69$$

Now let us verify the calculation of the discount rate according to expression (13). We will here show the calculation for $t = 0$. The rates at $t = 1$ and $t = 2$ can be determined accordingly. We start with calculating the factor v_1:

$$v_1 = \frac{100}{100 + 309.52} = 24.4186\ \%$$

We apply this factor together with r_f and r_C in expression (13) and obtain:

$$r_{\mathrm{CV},0} = \frac{(1 - 24.4186\ \%) \cdot 5\ \% + 24.4186\ \% \cdot 10\ \% + 10\ \% \cdot 5\ \%}{1 + 24.4186\ \% \cdot 5\ \% + (1 - 24.4186\ \%) \cdot 10\ \%} = 6.1785\ \%$$

Since the conditional expectations of the cash flows and values equal their unconditional expectations we can calculate the present value directly by the following calculation, which contains the time-varying discount rates shown in Figure 4:

$$V_0 = \frac{CF_1}{1 + r_{\mathrm{CV},0}} + \frac{CF_2}{(1 + r_{\mathrm{CV},0}) \cdot (1 + r_{\mathrm{CV},1})} + \frac{CF_3}{(1 + r_{\mathrm{CV},0}) \cdot (1 + r_{\mathrm{CV},1}) \cdot (1 + r_{\mathrm{CV},2})}$$

$$= \frac{100}{(1 + 6.1785\ \%)} + \frac{150}{(1 + 6.1785\ \%) \cdot (1 + 7.2028\ \%)} + \frac{200}{(1 + 6.1785\ \%) \cdot (1 + 7.2028\ \%) \cdot (1 + 10\ \%)}$$

$$= 385.69$$

A similar pattern of discount rates has been shown in Becker (2024) who studied the behavior of discount rates in the valuation of unlevered and levered firms. This and the previous section have analyzed cases where cash flows are not disrupted. The next section is devoted to the case where the stream of cash flows is discontinued.

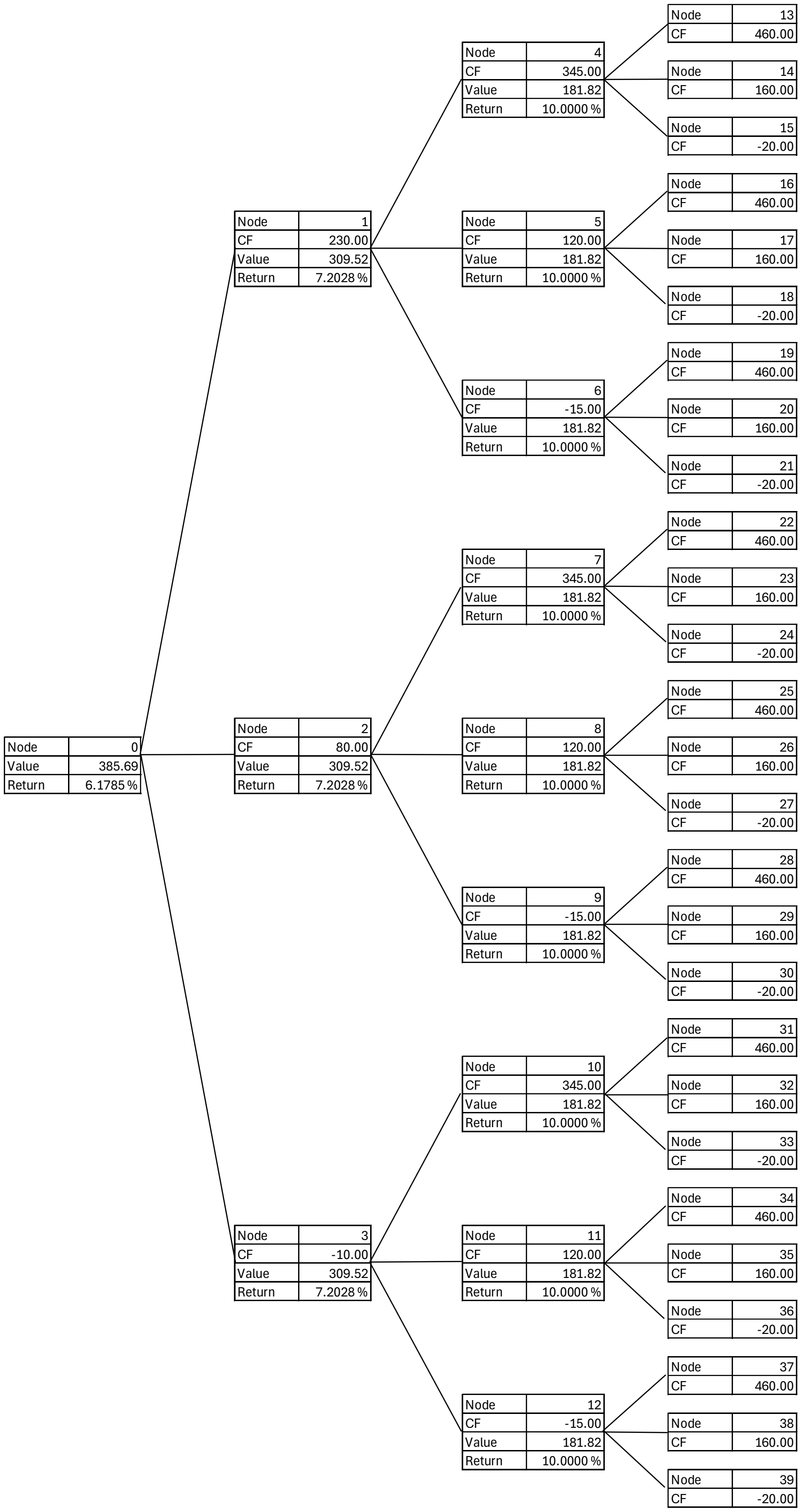

Figure 4: Evolution of cash flows, values, and discount rates in the non-disrupted trend-stationary case

5 Disrupted autoregressive cash flows

In this section, we turn back to an autoregressive cash flow like in section 3. However, we replace assumption 3a with an assumption that allows the discontinuation of the cash flow. The discontinuation of cash flows receives practical relevance in cases where firms declare bankruptcy, or investment projects can be abandoned if future prospects look bad.

Assumption 3c – Discontinued autoregressive cash flows: We assume the following process:

$$\tilde{z}_t = \begin{cases} z_{t-1} \cdot \tilde{\varepsilon}_t & \text{if} \quad z_{t-1} \geq 0 \\ 0 & \text{if} \quad z_{t-1} < 0 \end{cases}$$

Like before, $\tilde{\varepsilon}_t$ is a stochastic input-parameter that is drawn independently from the same time-invariant distribution $\mathcal{D}$. We assume some $z_0 > 0$ that initiates the process $\{\tilde{z}_t\}_{t=1,\dots,T}$ which we refer to as the basis cash flow. This cash flow is disrupted at point in time $t+1$ (set to Zero for the remaining lifetime) once this cash flow becomes negative in t. A scaling factor $a_t > 0$ allows the cash flow to be scaled up or down without changing its underlying stochasticity, i. e.:

$$\widetilde{CF}_t = a_t \cdot \tilde{z}_t$$

Using these relationships, we can formulate the autoregressive cash flow more directly such that we see how CF_t depends on CF_{t-1}:

$$\widetilde{CF}_t = \begin{cases} CF_{t-1} \cdot b_t \cdot \tilde{\varepsilon}_t & \text{if} \quad CF_{t-1} \geq 0 \\ 0 & \text{if} \quad CF_{t-1} < 0 \end{cases} \quad \text{where} \;\; b_t = \frac{a_t}{a_{t-1}}$$

For the analysis below, we reformulate this expression slightly:

$$\widetilde{CF}_t = \begin{cases} CF_{t-1} \cdot b_t \cdot \tilde{\varepsilon}_t & \text{if} \quad \varepsilon_{t-1} \geq 0 \\ 0 & \quad\quad \varepsilon_{t-1} < 0 \end{cases} \tag{14}$$

Here we exploit the fact that, whenever $\varepsilon_{t-k} \leq 0$ with $k > 1$ then $CF_{t-k+1} = CF_{t-k+2} = \cdots = 0$. This means that whenever the cash flow was disrupted at some point before $t-1$, the cash flow at $t-1$ is zero, even if $\varepsilon_{t-1} > 0$.

Figure 5 illustrates this case. Let us for example look at node 3. In this node the cash flow is negative. Therefore, the cash-flow stream is disrupted, and all future cash flows are set to zero. This implies that the continuation value in node 3 must be zero. We can furthermore observe that returns cannot be calculated in cases where both the expected cash flow and its value are zero (for example in nodes 10, 11, and 12). The input parameters used in the calculations are the same as in section 3. In what follows we develop the formulas that are necessary for obtaining the values and discount rates shown in Figure 5.

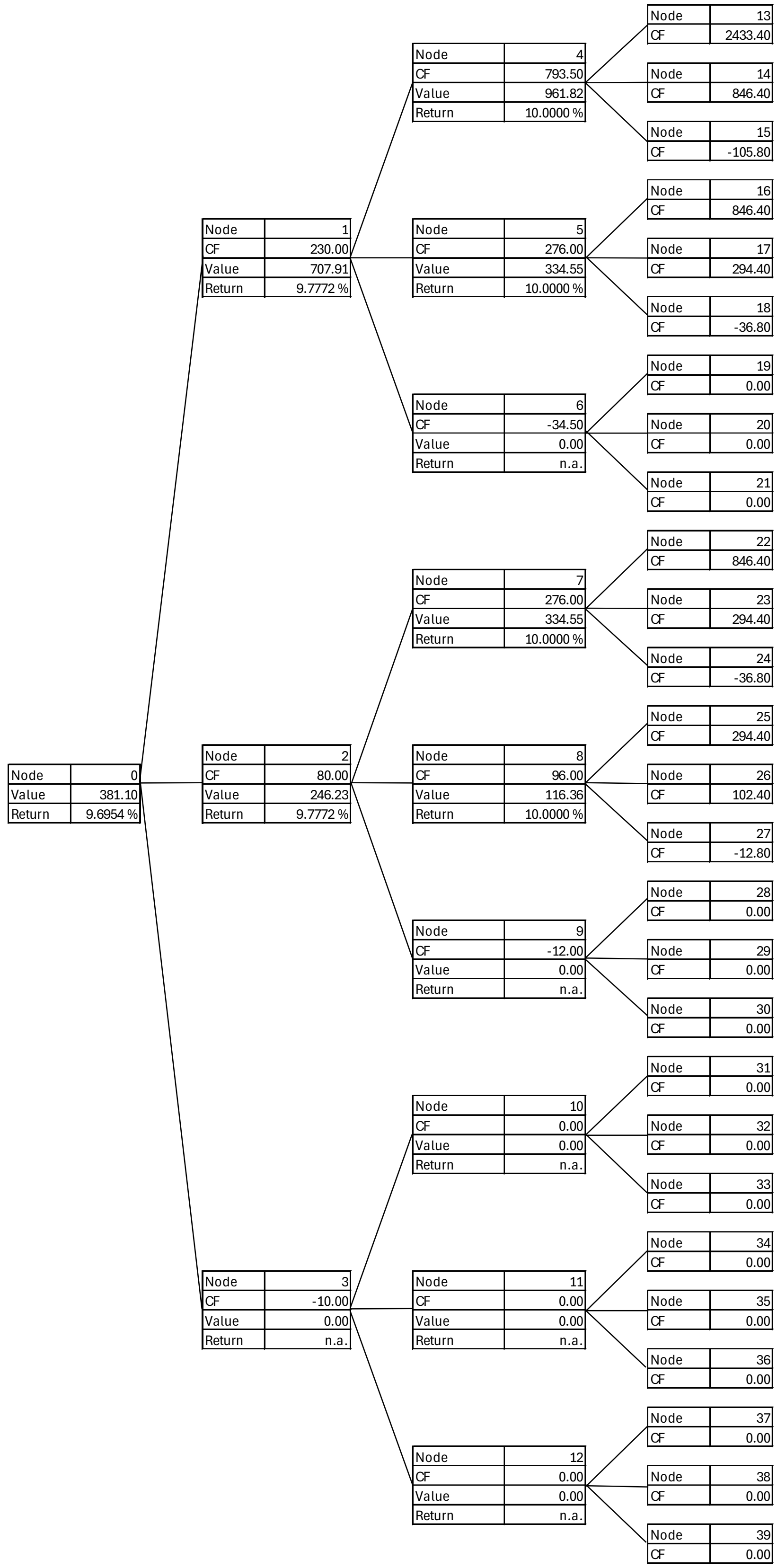

Figure 5: Evolution of cash flows, values, and discount rates in the disrupted autoregressive case

Let us have a look at the distribution of the values at point in time $t = 3$. For the first two child nodes (that appear in $t = 4$) of any parent node, we see that the futures (the remaining trees after these nodes) are the same, except for being scaled by the magnitude of the cash flow in the parent node. This means that the futures (i.e. all cash flows in the future) of the nodes 10, 11, 13, 14, 16, 17, and so forth, are proportional to each other. The third child node of any parent node has a value of 0 since the cash flow is discontinued.

At point in time $t = 2$, we have the same distribution of continuation values. The futures of the first two child nodes (for example 4 and 5, or 7 and 8) are the same except for being scaled by (proportional to) different initial cash flows, and the third child node (6 or 9) has a continuation value of 0.

This means that the distribution of the values in the first two child nodes of any parent node is completely determined by the distribution of the cash flow that appears in this parent node, while the third child node is zero. This distribution of the values in any $t + 1$ for a parent node in t is the same for all parent nodes in all points in time. Hence, the continuation values need to be discounted by the same discount rate. How the distribution of continuation values coincides with the distribution of the cash flows depends on the probability of discontinuation. The lower this probability, the closer will be the discount rate of the continuation values to the discount rate of the cash flows.

We now develop the valuation mathematically. We use the expression (14), and like before, we iterate backwards through time. We start with the calculation of the value at $T - 1$ (we neglect the scaling factors; they can be added back afterwards):

$$V_{T-1}\left(\widetilde{CF}_T\right) = \begin{cases} \dfrac{CF_{T-1} \cdot \mathbb{E}_{T-1}[\tilde{\varepsilon}_T]}{1 + r_{\mathrm{C}}} & \text{if} \quad \varepsilon_{T-1} \geq 0 \\ \square & \square \quad \square \\ 0 & \square \quad \varepsilon_{T-1} < 0 \end{cases}$$

All the deterministic parts are moved outside the expectation operator (note that we can also discount the Zero with $(1 + r_{\mathrm{C}})$ without changing the result). This yields:

$$V_{T-1}\left(\widetilde{CF}_T\right) = \frac{CF_{T-1}}{1 + r_{\mathrm{C}}} \cdot \begin{cases} \mathbb{E}_{T-1}[\tilde{\varepsilon}_T] & \text{if} \quad \varepsilon_{T-1} \geq 0 \\ \square & \square \quad \square \\ 0 & \square \quad \varepsilon_{T-1} < 0 \end{cases}$$

The cash flow CF_{T-1} depends on the cash flow CF_{T-2} and the realization of ε_{T-2} according to (14). Hence, we can write:

If $\varepsilon_{T-2} \geq 0$ then we have:

$$V_{T-1}\left(\widetilde{CF}_T\right) = \frac{CF_{T-2} \cdot \varepsilon_{T-1}}{1 + r_{\mathrm{C}}} \cdot \begin{cases} \mathbb{E}_{T-1}[\tilde{\varepsilon}_T] & \text{if} \quad \varepsilon_{T-1} \geq 0 \\ \square & \square \quad \square \\ 0 & \square \quad \varepsilon_{T-1} < 0 \end{cases}$$

If $\varepsilon_{T-2} < 0$ then we have:

$$V_{T-1}\left(\widetilde{CF}_T\right) = 0$$

Let us work on the first case where $\varepsilon_{T-2} \geq 0$. The expectation of V_{T-1} seen from $T-2$ is as follows:

$$\mathbb{E}_{T-2}[\tilde{V}_{T-1}] = \mathbb{E}_{T-2}\left[\frac{CF_{T-2} \cdot \tilde{\varepsilon}_{T-1}}{1+r_{\mathrm{C}}} \cdot \begin{cases} \mathbb{E}_{T-1}[\tilde{\varepsilon}_T] & \text{if} \quad \tilde{\varepsilon}_{T-1} \geq 0 \\ 0 & \quad\;\; \tilde{\varepsilon}_{T-1} < 0 \end{cases}\right]$$

Since any $\tilde{\varepsilon}_t$ is independently and identically distributed, $\mathbb{E}_{T-1}[\tilde{\varepsilon}_T]$ is deterministic; we denote it as $\mathbb{E}_{T-1}[\tilde{\varepsilon}_T] = \bar{\varepsilon}$.

$$\mathbb{E}_{T-2}[\tilde{V}_{T-1}] = \mathbb{E}_{T-2}\left[\frac{CF_{T-2} \cdot \tilde{\varepsilon}_{T-1}}{1+r_{\mathrm{C}}} \cdot \begin{cases} \bar{\varepsilon} & \text{if} \quad \tilde{\varepsilon}_{T-1} \geq 0 \\ 0 & \quad\;\; \tilde{\varepsilon}_{T-1} < 0 \end{cases}\right]$$

In this expression, we swap $\tilde{\varepsilon}_{T-1}$ and $\bar{\varepsilon}$ (note that Zero divided by $\bar{\varepsilon}$ and multiplied by $\tilde{\varepsilon}_{T-1}$ remains zero). Hence, we can rewrite:

$$\mathbb{E}_{T-2}[\tilde{V}_{T-1}] = \frac{CF_{T-2} \cdot \bar{\varepsilon}}{1+r_{\mathrm{C}}} \cdot \mathbb{E}_{T-2}\left[\begin{cases} \tilde{\varepsilon}_{T-1} & \text{if} \quad \tilde{\varepsilon}_{T-1} \geq 0 \\ 0 & \quad\;\; \tilde{\varepsilon}_{T-1} < 0 \end{cases}\right]$$

The stochastic term $\begin{cases} \tilde{\varepsilon}_{T-1} & \text{if} \quad \tilde{\varepsilon}_{T-1} \geq 0 \\ 0 & \quad\;\; \tilde{\varepsilon}_{T-1} < 0 \end{cases}$ has another risk than the term $\tilde{\varepsilon}_{T-1}$ (all the negative realizations in $\tilde{\varepsilon}_{T-1}$are replaced by zero). Hence, we need to apply another discount rate. Let us denote this rate as $r_{\mathrm{V},T-2}$. Let us furthermore denote the new stochastic term and its expectation as follows:

$$\tilde{\varepsilon}_{+} = \left\{\begin{matrix} \tilde{\varepsilon} & \text{if} & \tilde{\varepsilon} \geq 0 \\ 0 & & \tilde{\varepsilon} < 0 \end{matrix}\right\}, \qquad \bar{\varepsilon}_{+} = \mathbb{E}\left[\left\{\begin{matrix} \tilde{\varepsilon} & \text{if} & \varepsilon_t \geq 0 \\ 0 & & \tilde{\varepsilon} < 0 \end{matrix}\right\}\right]$$

Now we can calculate the value at $T-2$. In the following expression we consider that $\varepsilon_{T-2} \geq 0$ or $\varepsilon_{T-2} < 0$:

$$V_{T-2}[\tilde{V}_{T-1}] = \frac{CF_{T-2} \cdot \bar{\varepsilon} \cdot \bar{\varepsilon}_{+}}{(1+r_{\mathrm{C}}) \cdot \left(1+r_{\mathrm{V},T-2}\right)} \cdot \begin{cases} 1 & \text{if} \quad \varepsilon_{T-2} \geq 0 \\ 0 & \quad\;\; \varepsilon_{T-2} < 0 \end{cases}$$

Again, we look at the stochasticity and expectation of this term as seen from $T-3$, where we now need to consider $\widetilde{CF}_{T-2} = CF_{T-3} \cdot \tilde{\varepsilon}_{T-2}$ (if $\varepsilon_{T-3} \geq 0$):

$$\mathbb{E}_{T-3}[\tilde{V}_{T-2}] = \mathbb{E}_{T-3}\left[\frac{CF_{T-3} \cdot \tilde{\varepsilon}_{T-2} \cdot \bar{\varepsilon} \cdot \bar{\varepsilon}_{+}}{(1+r_{\mathrm{C}}) \cdot \left(1+r_{\mathrm{V},T-2}\right)} \cdot \left\{\begin{matrix} 1 & \text{if} & \tilde{\varepsilon}_{T-2} \geq 0 \\ 0 & & \tilde{\varepsilon}_{T-2} < 0 \end{matrix}\right\}\right]$$

We move all deterministic variables outside the expectation operator. This yields:

$$\mathbb{E}_{T-3}[\tilde{V}_{T-2}] = \frac{CF_{T-3} \cdot \bar{\varepsilon} \cdot \bar{\varepsilon}_{+}}{(1+r_{\mathrm{C}}) \cdot \left(1+r_{\mathrm{V},T-2}\right)} \cdot \mathbb{E}_{T-3}\left[\left\{\begin{matrix} \tilde{\varepsilon}_{T-2} & \text{if} & \tilde{\varepsilon}_{T-2} \geq 0 \\ 0 & & \tilde{\varepsilon}_{T-2} < 0 \end{matrix}\right\}\right]$$

The stochastic term that drives the discount rate, namely $\left\{\begin{matrix} \tilde{\varepsilon}_{T-2} & \text{if} & \tilde{\varepsilon}_{T-2} \geq 0 \\ 0 & & \tilde{\varepsilon}_{T-2} < 0 \end{matrix}\right\}$, is the same as above (note that $\tilde{\varepsilon}_t$ is independently and identically distributed). Therefore, it needs to be discounted with the same discount rate, i. e. $r_{\mathrm{V},T-2} = r_{\mathrm{V},T-3}$. We could continue the backward iteration to the beginning of the lifetime ($t = 0$) and would obtain the same stochastic factor $\tilde{\varepsilon}_+$ to which the continuation values are proportional. Hence, the discount rate to be applied to the continuation values is time invariant, and we use the notation r_V.

We can conclude two things:

(a) The expected cash flow to be obtained at any point in time $t = 2, \dots, T$ needs to be adjusted with some multiple of $\bar{\varepsilon}_+$. This is because we do not consider the full distribution of cash flows.

(b) The cash flow at some point in time t has to be discounted once with r_C to achieve its value at $t-1$, and afterwards this value is further discounted with the rate r_V.

The recursive valuation can therefore be stated as:

$$\mathbb{E}_0\left[V_{t-1}\left[\widetilde{CF}_t + \tilde{V}_t\right]\right] = \frac{\mathbb{E}_0\left[\widetilde{CF}_{t-1}\right] \cdot \bar{\varepsilon} \cdot \bar{\varepsilon}_+^{t-1}}{1 + r_\mathrm{C}} + \frac{\mathbb{E}_0[\tilde{V}_t]}{1 + r_\mathrm{V}} \tag{15}$$

where $\mathbb{E}_0$ indicates that we deal with unconditional expectations (expectations seen from the perspective at $t = 0$). The sum of the discounted cash flows can be written as:

$$V_0 = \sum_{t=1}^{T} \frac{\mathbb{E}_0\left[\widetilde{CF}_t\right] \cdot \bar{\varepsilon} \cdot \bar{\varepsilon}_+^{t-1}}{(1 + r_\mathrm{C}) \cdot (1 + r_\mathrm{V})^{t-1}} \tag{16}$$

In what follows, we show some of the numerical calculations for the example illustrated in Figure 5. Based on the same input parameters like in section 3, we can determine the expectations of $\tilde{\varepsilon}$ and $\tilde{\varepsilon}_+$:

$$\bar{\varepsilon} = \frac{1}{3} \cdot 2.3 + \frac{1}{3} \cdot 0.8 + \frac{1}{3} \cdot (-0.1) = 1, \qquad \bar{\varepsilon}_+ = \frac{1}{3} \cdot 2.3 + \frac{1}{3} \cdot 0.8 + \frac{1}{3} \cdot 0 = \frac{31}{30}$$

With these parameters, we can compute the adjusted expected cash flows:

$$\mathbb{E}_0[CF_3]' = \mathbb{E}_0[CF_3] \cdot \bar{\varepsilon} \cdot \bar{\varepsilon}_+^2 = 200 \cdot 1 \cdot (31/30)^2 = 213.5556$$

$$\mathbb{E}_0[CF_2]' = \mathbb{E}_0[CF_2] \cdot \bar{\varepsilon} \cdot \bar{\varepsilon}_+ = 150 \cdot 1 \cdot (31/30) = 155$$

$$\mathbb{E}_0[CF_1]' = \mathbb{E}_0[CF_1] \cdot \bar{\varepsilon} = 100 \cdot 1 = 100$$

Like before any stochastic variable proportional to $\tilde{\varepsilon}$ is discounted with $r_\mathrm{C} = 10\ \%$. In Addition, we need the discount rate that enables us to evaluate $\tilde{\varepsilon}_+$. Here, we apply the rate $r_\mathrm{V} = 9.6\ \%$. We chose this rate to be lower than r_C since we reduce the riskiness, when eliminating negative outcomes and replacing them with Zero.

The recursive valuation works as follows, if we discount parts of different risk separately:

$$\mathbb{E}_0[V_2] = \frac{\mathbb{E}_0[CF_3]'}{1 + r_\mathrm{C}} = \frac{213.5556}{1 + 10\ \%} = 194.1414$$

$$\mathbb{E}_0[V_1] = \frac{\mathbb{E}_0[CF_2]'}{1 + r_\mathrm{C}} + \frac{\mathbb{E}_0[V_2]}{1 + r_\mathrm{V}} = \frac{155}{1 + 10\ \%} + \frac{194.1414}{1 + 9.6\ \%} = 318.0454$$

$$V_0 = \frac{\mathbb{E}_0[CF_1]'}{1 + r_\mathrm{C}} + \frac{\mathbb{E}_0[V_1]}{1 + r_\mathrm{V}} = \frac{100}{1 + 10\ \%} + \frac{318.0454}{1 + 9.6\ \%} = 381.0965$$

The sum of the discounted expected cash flows gives the same result:

$$V_0 = \frac{\mathbb{E}_0[CF_1]'}{1 + r_\mathrm{C}} + \frac{\mathbb{E}_0[CF_2]'}{(1 + r_\mathrm{C}) \cdot (1 + r_\mathrm{V})} + \frac{\mathbb{E}_0[CF_3]'}{(1 + r_\mathrm{C}) \cdot (1 + r_\mathrm{V})^2}$$

$$= \frac{100}{(1 + 10\ \%)} + \frac{155}{(1 + 10\ \%) \cdot (1 + 9.6\ \%)} + \frac{213.5556}{(1 + 10\ \%) \cdot (1 + 9.6\ \%)^2}$$

$$= 381.0965$$

Another equivalent calculation is based on the discount rates to applied to the cash flow $\mathbb{E}_0[CF_t]'$ and continuation value $\mathbb{E}_0[V_t]$ jointly. They can be derived from the cash flows and values as follows:

$$r_{\mathrm{CV},2} = r_\mathrm{C} = 10\ \%$$

$$r_{\mathrm{CV},1} = \frac{\mathbb{E}_0[CF_2]' + \mathbb{E}_0[V_2]}{\mathbb{E}_0[V_1]} - 1 = \frac{155 + 194.1414}{318.0454} - 1 = 9.777\ \%$$

$$r_{\mathrm{CV},0} = \frac{\mathbb{E}_0[CF_1]' + \mathbb{E}_0[V_1]}{\mathbb{E}_0[V_0]} - 1 = \frac{100 + 318.0454}{381.0965} - 1 = 9.695\ \%$$

The direct calculation of the present value V_0 based on this time-variant discount rates is as follows:

$$V_0 = \frac{\mathbb{E}_0[CF_1]'}{1 + r_{\mathrm{CV},0}} + \frac{\mathbb{E}_0[CF_2]'}{\left(1 + r_{\mathrm{CV},0}\right) \cdot \left(1 + r_{\mathrm{CV},1}\right)} + \frac{\mathbb{E}_0[CF_3]'}{\left(1 + r_{\mathrm{CV},0}\right) \cdot \left(1 + r_{\mathrm{CV},1}\right) \cdot \left(1 + r_{\mathrm{CV},2}\right)}$$

$$= \frac{100}{(1 + 9.695\ \%)} + \frac{155}{(1 + 9.695\ \%) \cdot (1 + 9.777\ \%)} + \frac{213.5556}{(1 + 9.695\ \%) \cdot (1 + 9.777\ \%) \cdot (1 + 10\ \%)}$$

$$= 381.0965$$

It is worth mentioning that the formulas and discussion above rest on the assumption that the last negative cash flow, the cash flow just before the cash-flow stream is incurred (it is part of the valuation). Alternatively, we could have assumed that whenever a negative cash flow appears it will not be paid (incurred) as it would be the case if the project owners or equity holders have limited liability. In other words, the stream of cash flows is immediately disrupted before a negative cash flow can appear.

In this case, the formulas (15) and (16) can be adjusted by replacing $\bar{\varepsilon}$ with $\bar{\varepsilon}_+$. Hence, these formulas become the following:

Recursive valuation

$$\mathbb{E}_0\left[V_{t-1}\left[\widetilde{CF}_t + \tilde{V}_t\right]\right] = \frac{\mathbb{E}_0\left[\widetilde{CF}_{t-1}\right] \cdot \bar{\varepsilon}_+^t}{1 + r_\mathrm{C}} + \frac{\mathbb{E}_0\left[\tilde{V}_t\right]}{1 + r_\mathrm{V}} \tag{17}$$

Sum of the discounted cash flows

$$V_0 = \sum_{t=1}^{T} \frac{\mathbb{E}_0\left[\widetilde{CF}_t\right] \cdot \bar{\varepsilon}_+^t}{(1 + r_\mathrm{C}) \cdot (1 + r_\mathrm{V})^{t-1}} \tag{18}$$

6 Alternative rules for the disruption of cash flows

In this section, we consider another, in our opinion, practically more appealing rule for disrupting cash-flow streams. In section 5, the stream of cash flows was discontinued whenever we observed a negative cash flow. In practice, however, we would like to continue projects with temporary negative cash flows as long as they promise a positive value of cash flows in the future. Hence, it is the future value that determines whether projects are abandoned or not. For example, a firm with limited liability, could declare bankruptcy (or abandon an investment) at point in time t whenever both $CF_t < 0$ and $CF_t + V_t < 0$. This means, whenever the firm receives a negative cash flow, it considers whether this cash flow can be compensated by some larger and positive value of future cash flows. If this is not the case the project or firm owners let the project/firm go bankrupt immediately, without taking responsibility for the negative cash flow (by not paying suppliers or debt holders). Another possibility is to consider a company with unlimited liability. It will cease its business if the prospects look poor (i. e. $V_t < 0$), but it is responsible for the negative cash flow that has arisen so far. In the following, we restrict our analysis to the following rule:

Assumption 3d - Autoregressive cash flows with discontinuation triggered by negative continuation value: The basis process is given as follows:

$$\tilde{z}_t = z_{t-1} \cdot \tilde{\varepsilon}_t$$

Like before, $\tilde{\varepsilon}_t$ is a stochastic input-parameter that is drawn independently from the same time-invariant distribution $\mathcal{D}$, and $\{\tilde{z}_t\}_{t=1,\dots,T}$ is the basis process responsible for the cash flow. We allow the cash flow to be scaled:

$$\widetilde{CF}_t = a_t \cdot \tilde{z}_t$$

This represents the cash flow, as if the firm (stream of cash flows) would continue into the future without disruption. The following terms/rules define the cash flow and continuation value for the case of disruption:

$$\widetilde{CF}'_t = \begin{cases} \widetilde{CF}_t & \text{if} \quad \widetilde{CF}_t + \tilde{V}_t \geq 0 \\ 0 & \text{if} \quad \widetilde{CF}_t + \tilde{V}_t < 0 \end{cases}$$

$$\tilde{V}'_t = \begin{cases} \tilde{V}_t & \text{if} \quad \widetilde{CF}_t + \tilde{V}_t \geq 0 \\ 0 & \text{if} \quad \widetilde{CF}_t + \tilde{V}_t < 0 \end{cases}$$

It is important to notice that we first need to generate the whole sequence $\{\widetilde{CF}_t\}_{t=1,\dots,T}$ before we can determine the $V_{t-1}(\widetilde{CF}'_t + \tilde{V}'_t)$ by means of backward iteration. Using the same input parameters for $\tilde{\varepsilon}_t$ and a_t like in section 3, we obtain the cash flows shown in Figure 6.

The backward iteration by means of discount rates can be algebraically challenging. We therefore use the following backward iteration based on risk neutral probabilities:

$$V_{[n]} = \sum_{m \in \mathcal{C}(n)} \frac{\pi_{[m]}}{1 + r_{\mathrm{f}}} \cdot \left(CF_{[m]} + V_{[m]} \right) \quad \text{for all nodes } n \text{ at points } t = 0 \text{ to } t = T - 1 \tag{19}$$

where $\pi_{[m]}$ represents the risk-neutral probability of transitioning to node (state) m, and $\mathcal{C}(n)$ is the set of child nodes that arise from parent node n.

Let us explain some of the calculations behind Figure 6. The cash flows generated throughout the tree are the same as in Figure 2. For every node in Figure 6 it is also indicated whether the cash flow will cease (stop) if we observe the condition $\widetilde{CF}_t + \tilde{V}_t < 0$. Let us, for example, look at the cash flows in nodes 19, 20, and 21, which are –105.80, –36.8, and 4.6, respectively. Due to the limited liability, the investment owner cannot be held responsible for the negative cash flows in nodes 19 and 20. Hence, we have $\mathbb{E}_{[6]}\left[\widetilde{CF}_3\right]' = \frac{1}{3} \cdot 0 + \frac{1}{3} \cdot 0 + \frac{1}{3} \cdot 4.6 \approx 1.5333$ (not shown in the figure). The value of this expected cash flow is $V_{[6]}\left[\widetilde{CF}_3\right]' = \frac{\pi_3 \cdot 4.6}{1+5\,\%} = \frac{0.354180 \cdot 4.6}{1+5\,\%} = 1.5516$. This value is larger than the expected cash flow, which indicates a negative discount rate. The amount of the negative cash flow in node 6 cannot be compensated by this value. Hence, the investment owners will abandon the project already in $t = 2$ (if node 6 is realized as the state of the world). Therefore, the value to be considered in further backward iteration is $\tilde{V}'_{[6]} = 0$. The same logic is applied to all other nodes in the tree. Having calculated the values of all nodes, the discount rates can be computed:

$$r_{[n]} = \frac{\sum_{m \in \mathcal{C}(n)} p_{[m]} \cdot \left(CF_{[m]} + V_{[m]} \right)}{V_{[n]}} - 1$$

where $p_{[m]}$ represents the real probability of transitioning to node (state) m, and $\mathcal{C}(n)$ is the set of child nodes that arise from parent node n.

We recognize that the discount rates in all nodes where the investment is continued are the same, and they coincide with $r_{\mathrm{V}} = 9.6\,\%$. We also see that the stream of cash flows is disrupted in the same nodes as in section 5. However, contrary to the case described in section 5, negative cash flows are not realized.

For the example in Figure 6, this implies that we can apply $r_{\mathrm{V}} = r_{\mathrm{C}} = r_{\mathrm{CV}} = 9.6\,\%$ to both continuation values and cash flows. Therefore, the following valuation is applicable for the example:

$$V_{t-k}\left[\widetilde{CF}_t\right] = \begin{cases} \dfrac{CF_{t-k} \cdot \bar{\varepsilon}_{+}^{k}}{(1 + r_{\mathrm{CV}})^k} & \text{if} \quad \varepsilon_{t-k} \geq 0 \\ 0 & \quad\quad \varepsilon_{t-k} < 0 \end{cases}$$

This valuation is the same as expression (15), only that now $r_{\mathrm{V}} = r_{\mathrm{C}}$. Equivalently the sum of discounted cash flows can be written as:

$$V_0\left[\left\{\widetilde{CF}_1, \ldots, \widetilde{CF}_1\right\}\right] = \sum_{t=1}^{T} \frac{CF_t \cdot \bar{\varepsilon}_{+}^{t}}{(1 + r_{\mathrm{V}})^t}$$

Filling this expression with numerical values, we obtain the value at $t = 0$ as follows:

$$V_0\left[\{\widetilde{CF}_1, \dots, \widetilde{CF}_1\}\right] = \frac{100}{1 + 9.6\,\%} + \frac{150 \cdot (31/30)}{(1 + 9.6\,\%)^2} + \frac{200 \cdot (31/30)^3}{(1 + 9.6\,\%)^3} \approx 395.24$$

Again, in this example we observe that the required return is deterministic and constant throughout time, because the continuation values become zero whenever the stochastic parameter $\tilde{\varepsilon}_t$ is negative. However, there are situations where this is not the case. The longer the lifetime of the cash-flow stream the larger the continuation value. Particularly, the rule $\widetilde{CF}_t + \tilde{V}_t < 0$ prefers positive values to be propagated to the present, while negative values are not. This implies that positive continuation values eventually outperform (overcompensate) negative cash flows. Such a situation can also be created if we scale future cash flows up compared to present cash flows. Such a situation is shown in Figure 7. In this figure, the initial stream of expected cash flows is $\mathbb{E}_0\left[\widetilde{CF}_1\right] = 100$, $\mathbb{E}_0\left[\widetilde{CF}_2\right] = 150$, and $\mathbb{E}_0\left[\widetilde{CF}_3\right] = 10{,}000$. Basically, the last cash flow is scaled up tremendously, such that we can show the effects in a scenario tree with few points in time. In Figure 7 we recognize that discount rates now become stochastic and time-variant. Hence, the traditional present-value approach cannot be applied. Particularly, we cannot discount an expected cash flow $\mathbb{E}_0\left[\widetilde{CF}_t\right]$ with an expected discount rate $\mathbb{E}_0[\tilde{r}_{t-1}]$. Instead, we require a stochastic backward iteration of the following form, where we would have to estimate different discount rates for different states in the world:

$$V_{[n]} = \sum_{m \in \mathcal{C}(n)} \frac{p_{[m]}}{1 + r_{[\mathrm{n}]}} \cdot \left(CF_{[m]} + V_{[m]}\right) \quad \text{for all nodes } n \text{ at points } t = 0 \text{ to } t = T - 1 \qquad (20)$$

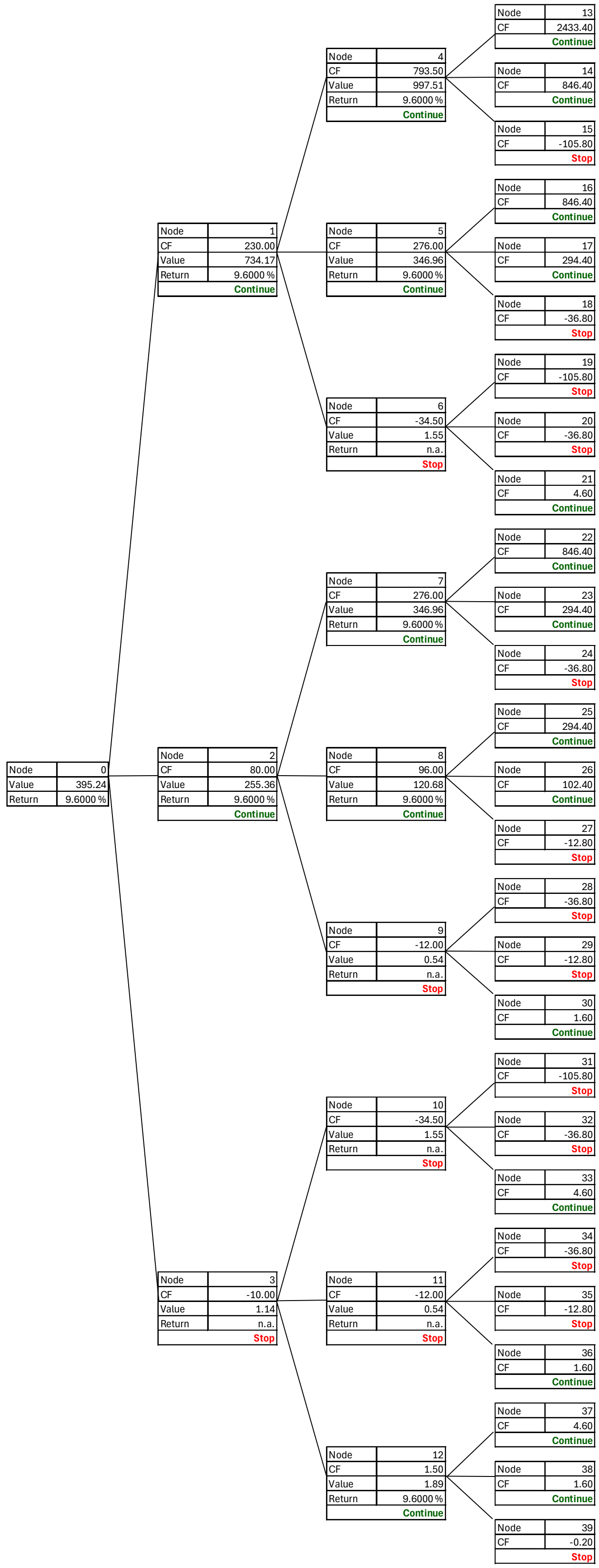

Figure 6: Evolution of cash flows, values, and discount rates in the disrupted autoregressive case with the rule of disruption: $\boldsymbol{V + CF < 0}$

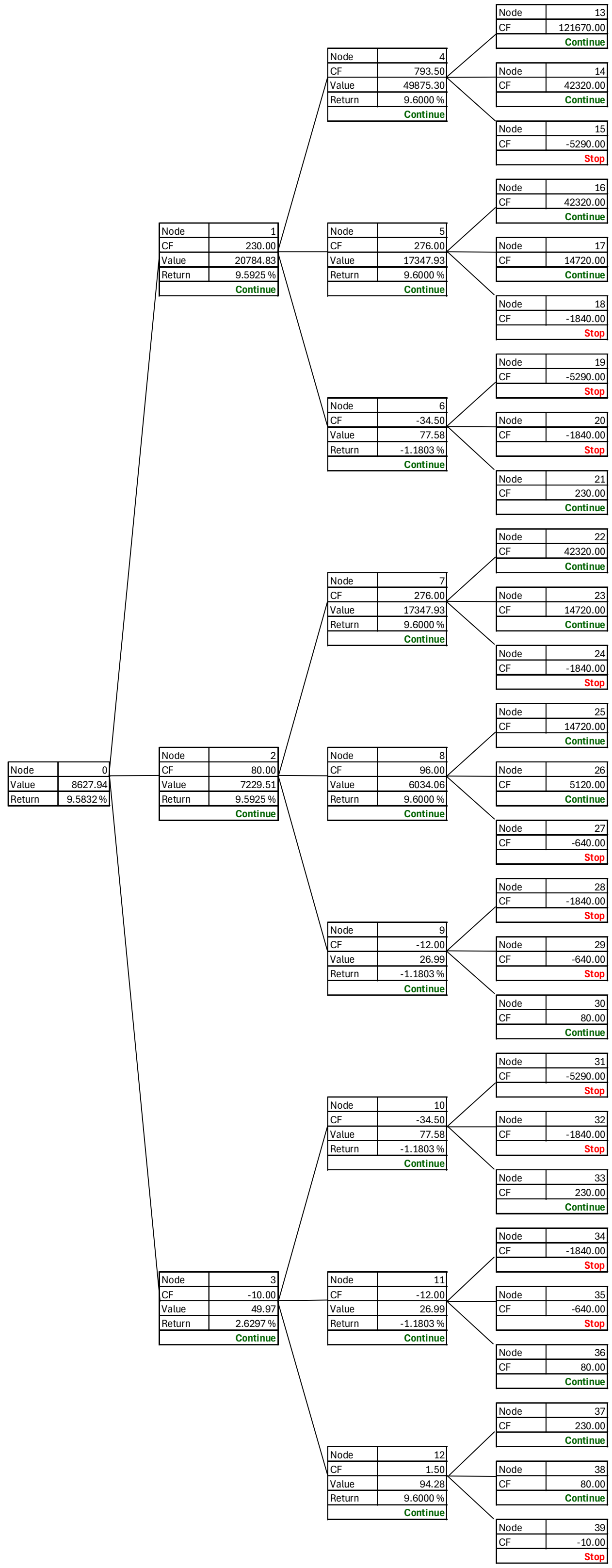

Figure 7: Stochastic discount rates for disrupted autoregressive CF streams

7 Summary and Outlook

The purpose of this paper was to examine the behavior of the required return in the present value method dependent on the stochastic properties of the stream of cash flows. Particularly we studied the following cases: (1) first-order auto regressive process without disruption of cash flows, (2) trend-stationary process without disruption of cash flows, (3) first-order auto regressive process with an disruption in case of negative cash flows, and (4) first-order auto regressive process with disruption in case of negative future values. Table 2 summarizes the formulas and most important properties for these cases.

Assuming a time-invariant pricing kernel, required returns (or discount rates) are deterministic and time-invariant if the cash-flow generating process is undisrupted and autoregressive of first order. Other processes generally lead to either time-varying or stochastic discount rates. If the cash flow is generated by a trend-stationary process or particular types of disrupted autoregressive processes, the present value approach can easily be adjusted to produce correct present values. "Correct" is here meant in the sense that we obey the arbitrage-freeness of markets, and that discount rates (required returns) reflect the opportunity costs of capital. However, more complex cash flows generally require stochastic discount rates. This means that the traditional present value approach is not applicable.

Of course, from a pragmatic point of view, one could always claim that there is some constant yield (discount rate), that leads to the same present value as if stochastic discount rates were applied. Let us for example look at Figure 7. The numerical values in this figure our based on the following initial expected cash flow: $CF_1 = 100$, $CF_2 = 150$ and $CF_3 = 2{,}000$. The present value $V_0 = 8{,}627.94$ was computed by the risk-neutral approach. The same present value could have been achieved by using a constant and time-invariant discount rate of $r = 5.984\ \%$. It would produce exactly the same present value at $t = 0$ like in Figure 7. However, the question remains as to what economic meaning should be attributed to this discount rate. This rate would only compute a correct present value for $t = 0$, but not for any other point in time. It cannot be interpreted as an opportunity cost of capital. If we cannot attribute such a meaning to the discount rate, the question arises how it should be determined empirically.

In the very end, we can say that traditional present value methods work well if we assume autoregressive processes. As long as the valuation practitioner is aware of this assumption and can live with this assumption, there is no need to think about alternative valuation methods. For more complicated cash flows, the valuation practitioner needs to ask, whether the present value method should be replaced by approaches that use stochastic discount factors or risk neutral probabilities. This is a common approach with respect to the valuation of financial or real options, and there exists empirical research that aims at estimating these quantities (pricing kernels). Traditional present value calculations are very useful with respect to banking services like savings accounts, mortgages or reversed mortgages, where we basically work with deterministic cash flows and interest rates. However, these traditional calculations may not be applicable in stochastic environments. Standard textbooks, however, provide the impression that standard present value calculations can easily be applied to capital budgeting, investment analysis, or firm valuation. We believe it is important to point out the limitations and strict assumptions of this approach early in the education of finance students and we hope that this paper shows a step in this direction.

(A) Nondisrupted first-order autoregressive cash flow:	*Recursive valuation:* $V_{t-1} = \frac{\mathbb{E}_0[\widetilde{CF}_t] + \mathbb{E}_0[\tilde{V}_t]}{1 + r_{\mathrm{CV}}} = \frac{\mathbb{E}_0[\widetilde{CF}_t]}{1 + r_{\mathrm{C}}} + \frac{\mathbb{E}_0[\tilde{V}_t]}{1 + r_{\mathrm{V}}}$ where $r_{\mathrm{CV}} = r_{\mathrm{C}} = r_{\mathrm{V}}$ are constant. *Sum of discounted cash flows:* $V_0 = \sum_{t=1}^{T} \frac{\mathbb{E}_0[\widetilde{CF}_t]}{1 + r_{\mathrm{CV}}}$
(B) Nondisrupted trend-stationary cash flow	*Recursive valuation:* $V_{t-1} = \frac{\mathbb{E}_0[\widetilde{CF}_t] + V_t}{1 + r_{\mathrm{CV},t}} = \frac{\mathbb{E}_0[\widetilde{CF}_t]}{1 + r_{\mathrm{C}}} + \frac{V_t}{1 + r_{\mathrm{f}}}$ where $r_{\mathrm{CV},t} \neq r_{\mathrm{C}} \neq r_{\mathrm{V}}$ and $r_{\mathrm{V}} = r_{\mathrm{f}}$ usually, we would assume that $r_{\mathrm{C}} > r_{\mathrm{CV}} > r_{\mathrm{f}}$ *Sum of discounted cash flows:* $V_0 = \sum_{t=1}^{T} \frac{\mathbb{E}_0[\widetilde{CF}_t]}{(1 + r_{\mathrm{C}}) \cdot (1 + r_{\mathrm{f}})^{t-1}}$
(C) First-order autoregressive cash flow with rule of disruption $\varepsilon_t < 0$ (negative cash flows incur before disruption)	*Recursive valuation:* $\mathbb{E}_0[V_{t-1}] = \frac{\mathbb{E}_0[\widetilde{CF}_t] \cdot \bar{\varepsilon} \cdot \bar{\varepsilon}_+^{t-1} + \mathbb{E}_0[\tilde{V}_t]}{1 + r_{\mathrm{CV},t}} = \frac{\mathbb{E}_0[\widetilde{CF}_t] \cdot \bar{\varepsilon} \cdot \bar{\varepsilon}_+^{t-1}}{1 + r_{\mathrm{C}}} + \frac{\mathbb{E}_0[\tilde{V}_t]}{1 + r_{\mathrm{V}}}$ where $r_{\mathrm{CV},t} \neq r_{\mathrm{C}} \neq r_{\mathrm{V}}$ we may assume that $r_{\mathrm{C}} > r_{\mathrm{CV}} > r_{\mathrm{V}} > r_{\mathrm{f}}$ *Sum of discounted cash flows:* $V_0 = \sum_{t=1}^{T} \frac{\mathbb{E}_0[\widetilde{CF}_t] \cdot \bar{\varepsilon} \cdot \bar{\varepsilon}_+^{t-1}}{(1 + r_{\mathrm{C}}) \cdot (1 + r_{\mathrm{V}})^{t-1}}$
(D) First-order autoregressive cash flow with rule of disruption $\varepsilon_t < 0$ (negative cash flows are avoided before disruption)	*Recursive valuation:* $V_{t-1} = \frac{\mathbb{E}_0[\widetilde{CF}_t] \cdot \bar{\varepsilon}_+^{t} + \mathbb{E}_0[\tilde{V}_t]}{1 + r_{\mathrm{CV}}} = \frac{\mathbb{E}_0[\widetilde{CF}_t] \cdot \bar{\varepsilon}_+^{t}}{1 + r_{\mathrm{C}}} + \frac{\mathbb{E}_0[\tilde{V}_t]}{1 + r_{\mathrm{V}}}$ where $r_{\mathrm{CV}} = r_{\mathrm{C}} = r_{\mathrm{V}} > r_{\mathrm{f}}$ *Sum of discounted cash flows:* $V_0 = \sum_{t=1}^{T} \frac{\mathbb{E}_0[\widetilde{CF}_t] \cdot \bar{\varepsilon}_+^{t}}{(1 + r_{\mathrm{V}})^{t}}$
(E) First-order autoregressive cash flow with rule of disruption: $\widetilde{CF}_t + \tilde{V}_t < 0$	Special cases (A) or (D) can occur. Generally, the formulas above are not applicable. Generally, discount rates are time varying and stochastic.

Table 2: Present value formulas for different stochastic cash flow

Appendix A Forward generation of cash flows and backward iteration based on risk-neutral approach

We have used the risk-neutral evaluation to back up the values in the scenario trees of Figures 2 to 5. Furthermore, we used this approach for finding the values in the trees in Figure 6 and 7. This approach works as follows. First, all cash flows in all nodes are simulated in a forward manner. Specifically, this forward generation works as follows:

Create root node with node index $n = 0$
Set the auxiliary node index $m = 0$
Create a set $\mathcal{B}$ that contains node index $m = 0$
For each t from 0 to $T - 1$:
    For each node n at point in time t
        For node n, create an empty set $\mathcal{C}(n)$ to which the indexes of the child nodes will be added.
        For each state $s \in S$:
            Increase auxiliary node index by 1, i. e. $m = m + 1$
            Add node index m to $\mathcal{C}(n)$
            Generate corresponding $CF_{[m]}$ by the stochastic rule.
            For example:
            $CF_{[m]} = \varepsilon_s \cdot a_t$ (trend-stationary)

            or $CF_{[m]} = CF_{P(m)} \cdot \varepsilon_s \cdot b_t$ (auto-regressive)

            or $CF_{[m]} = \begin{cases} CF_{P(m)} \cdot \varepsilon_s \cdot a_t & \text{if} \quad \varepsilon_s \geq 0 \\ \square & \square \quad \square \\ 0 & \text{if} \quad \varepsilon_s < 0 \end{cases}$ (auto-regressive with disruption)

            Assign real and risk-neutral and probabilities to nodes:

$$p_{[m]} = p_s$$
$$\pi_{[m]} = \pi_s$$

            As long as $t = T - 2$, add node index m to the set $\mathcal{B}$.

After having generated the tree of cash flows, we can find all continuation values in all nodes at $t = 1$ to $t = T - 1$ and the present value at $t = 0$ by the following backward iteration:

For each node index n in $\mathcal{B}$, descending from the largest index to the smallest index:

Compute the continuation value:

$$V_{[n]} = \sum_{m \in \mathcal{C}(n)} \frac{\pi_{[m]}}{1 + r_\mathrm{f}} \cdot \left(CF_{[m]} + V_{[m]}\right)$$

Calculate the return:

$$r_{[n]} = \frac{\sum_{m \in \mathcal{C}(n)} p_{[m]} \cdot \left(CF_{[m]} + V_{[m]}\right)}{V_{[n]}} - 1$$

The forward generation of cash flows and the backward iteration requires the following input parameters: risk-free rate r_f, states $s = 1, \dots, S$, points in time $t = 1, \dots, T$, risk-neutral probabilities π_s, real probabilities p_s, stochastic parameter ε_s, scaling factors a_t or b_t

Appendix B Choice of discount rates and risk-neutral probabilities

In sections above we have assumed certain values for the discount rates. Particularly, these were $r_\mathrm{f} = 5\ \%$, $r_\mathrm{C} = 10\ \%$, and $r_\mathrm{V} = 9.6\ \%$. These rates were assumed. However, these discount rates and the pricing kernel are related to each other. With three scenarios, each with real probability 1/3, we can retrieve the rates $r_\mathrm{C} = 10\ \%$ and $r_\mathrm{V} = 9.6\ \%$ as follows:

For r_C, the discount rate of anything proportional to $\tilde{\varepsilon} = \{2.3, 0.8, -0.1\}$, we formulate:

$$\frac{\frac{1}{3} \cdot 2.3 + \frac{1}{3} \cdot 0.8 + \frac{1}{3} \cdot (-0.1)}{1 + 10\ \%} = \frac{\pi_1 \cdot 2.3 + \pi_2 \cdot 0.8 + (1 - \pi_1 - \pi_2) \cdot (-0.1)}{1 + 5\ \%}$$

where π_1, π_2 and $\pi_3 = (1 - \pi_1 - \pi_2)$ are the risk-neutral probabilities. This expression can be shortened:

$$\frac{1 + 5\ \%}{1 + 10\ \%} + 0.1 = \pi_1 \cdot 2.4 + \pi_2 \cdot 0.9$$

For r_V, the discount rate of $\tilde{\varepsilon} = \{2.3, 0.8, 0\}$ we formulate:

$$\frac{\frac{1}{3} \cdot 2.3 + \frac{1}{3} \cdot 0.8 + \frac{1}{3} \cdot 0}{1 + 9.6\ \%} = \frac{\pi_1 \cdot 2.3 + \pi_2 \cdot 0.8 + (1 - \pi_1 - \pi_2) \cdot 0}{1 + 5\ \%}$$

This is:

$$\frac{31/30 \cdot (1 + 5\ \%)}{1 + 9.6\ \%} = \pi_1 \cdot 2.3 + \pi_2 \cdot 0.8$$

If we solve these two equations, we obtain the risk-neutral probabilities:

$$\pi_1 \approx 0.315539, \qquad \pi_2 \approx 0.330281, \qquad \pi_3 \approx 0.354180$$

We can now use these risk-neutral probabilities to find values of other stochastic cash flows. The rate applied in Figure 3 is found as:

$$r_{\text{C (Figure 3)}} = \frac{p_1 \cdot 2.3 + p_2 \cdot 0.8 + (1 - p_1 - p_2) \cdot 0}{\pi_1 \cdot 2.3 + \pi_2 \cdot 0.8 + (1 - \pi_1 - \pi_2) \cdot 0} \cdot (1 + 5\ \%) - 1$$

$$= \frac{\frac{1}{3} \cdot 2.3 + \frac{1}{3} \cdot 0.8}{0.315539 \cdot 2.3 + 0.330281 \cdot 0.8} \cdot (1 + 5\ \%) - 1$$

$$= 9.6\ \%$$

Appendix C Trend-stationary case with disrupted cash flows

This appendix shows the calculation of present values and the behavior of discount rates for cash flows which are trend-stationary and disrupted according to the following assumption:

Assumption 3e – Disrupted trend-stationary cash flows: The stream of cash flows is driven by the following process:

$$\widetilde{CF}_t = \begin{cases} \tilde{\varepsilon}_t \cdot a_t & \text{if} \quad CF_{t-1} \geq 0 \\ 0 & \text{if} \quad CF_{t-1} < 0 \end{cases} \quad \text{with} \quad \mathbb{E}_{t-1}[\tilde{\varepsilon}_t] = 1 \tag{21}$$

In this process, we should enforce $a_t > 0$. Otherwise, the cash flow stops whenever $a_t < 0$.

By means of backward iteration, we compute the values:

The value at $T-1$:

$$V_{T-1} = \begin{cases} \dfrac{a_T \cdot \mathbb{E}_{T-1}[\tilde{\varepsilon}_T]}{1+r_C} & \text{if} \quad CF_{T-1} \geq 0 \\ 0 & \text{if} \quad CF_{T-1} < 0 \end{cases}$$

The expectation of this value seen from $T-2$ is as follows (note that $\mathbb{E}_{T-1}[\tilde{\varepsilon}_T] = 1$):

$$\mathbb{E}_{T-2}[\tilde{V}_{T-1}] = \frac{a_T}{1+r_C} \cdot \mathbb{E}_{T-2}\left[\begin{Bmatrix} 1 & \text{if} & a_{T-1} \cdot \tilde{\varepsilon}_{T-1} \geq 0 \\ 0 & \text{if} & a_{T-1} \cdot \tilde{\varepsilon}_{T-1} < 0 \end{Bmatrix}\right]$$

Because $\tilde{\varepsilon}_t$ is independently and identically distributed, the term $\begin{Bmatrix} 1 & \text{if} & a_{T-1} \cdot \tilde{\varepsilon}_{T-1} \geq 0 \\ 0 & \text{if} & a_{T-1} \cdot \tilde{\varepsilon}_{T-1} < 0 \end{Bmatrix}$ is independently and identically distributed for $a_t > 0$. We denote its expectation by φ_+. It needs to have its own discount rate r_V (note that this rate does not have the same magnitude/value as in the case of an autoregressive process). This discount rate becomes closer to the risk-free rate r_f as the probability of $\tilde{\varepsilon}_{T-1} < 0$ decreases.

The value at $T-2$ is:

$$V_{T-2}\left[\widetilde{CF}_{T-1} + \tilde{V}_{T-1}\right] = \begin{cases} \dfrac{\mathbb{E}_{T-2}\left[\widetilde{CF}_{T-1}\right]}{1+r_C} + \dfrac{a_T \cdot \varphi_+}{(1+r_C) \cdot (1+r_V)} & \text{if} \quad CF_{T-2} \geq 0 \\ 0 & \text{if} \quad CF_{T-2} < 0 \end{cases}$$

This is:

$$V_{T-2}\left[\widetilde{CF}_{T-1} + \tilde{V}_{T-1}\right] = \left(\frac{a_{T-1}}{1+r_C} + \frac{a_T \cdot \varphi_+}{(1+r_C) \cdot (1+r_V)}\right) \cdot \begin{cases} 1 & \text{if} \quad a_{T-2} \cdot \tilde{\varepsilon}_{T-2} \geq 0 \\ 0 & \text{if} \quad a_{T-2} \cdot \tilde{\varepsilon}_{T-2} < 0 \end{cases}$$

The expectation at $T-3$:

$$\mathbb{E}_{T-3}[\tilde{V}_{T-2}] = \frac{a_{T-1} \cdot \varphi_+}{1 + r_\mathrm{C}} + \frac{a_T \cdot \varphi_+^2}{(1 + r_\mathrm{C}) \cdot (1 + r_\mathrm{V})}$$

Hence, the value at $T-3$ is:

$$V_{T-3}\left[\widetilde{CF}_{T-2} + \tilde{V}_{T-2}\right] = \frac{a_{T-2}}{1 + r_\mathrm{C}} + \frac{a_{T-1} \cdot \varphi_+}{(1 + r_\mathrm{C}) \cdot (1 + r_\mathrm{V})} + \frac{a_T \cdot \varphi_+^2}{(1 + r_\mathrm{C}) \cdot (1 + r_\mathrm{V})^2}$$

Recognizing the recursive structure, we can state the value at $t = 0$ as follows:

$$V_0 = \sum_{t=1}^{T} \frac{a_t \cdot \varphi_+^{(t-1)}}{(1 + r_\mathrm{C}) \cdot (1 + r_\mathrm{V})^{(t-1)}}$$

With numerical values this is:

$$\varphi_+ = \frac{1}{3} \cdot 1 + \frac{1}{3} \cdot 1 + \frac{1}{3} \cdot 0 = 2/3$$

$$V_0 = \frac{100}{1 + 10\,\%} + \frac{150 \cdot 2/3}{(1 + 10\,\%) \cdot (1 + 8.3894\,\%)} + \frac{200 \cdot (2/3)^2}{(1 + 10\,\%) \cdot (1 + 8.3894\,\%)} = 243.5647$$

The behavior of cash flows, discount rates, and values is shown in Figure 8.

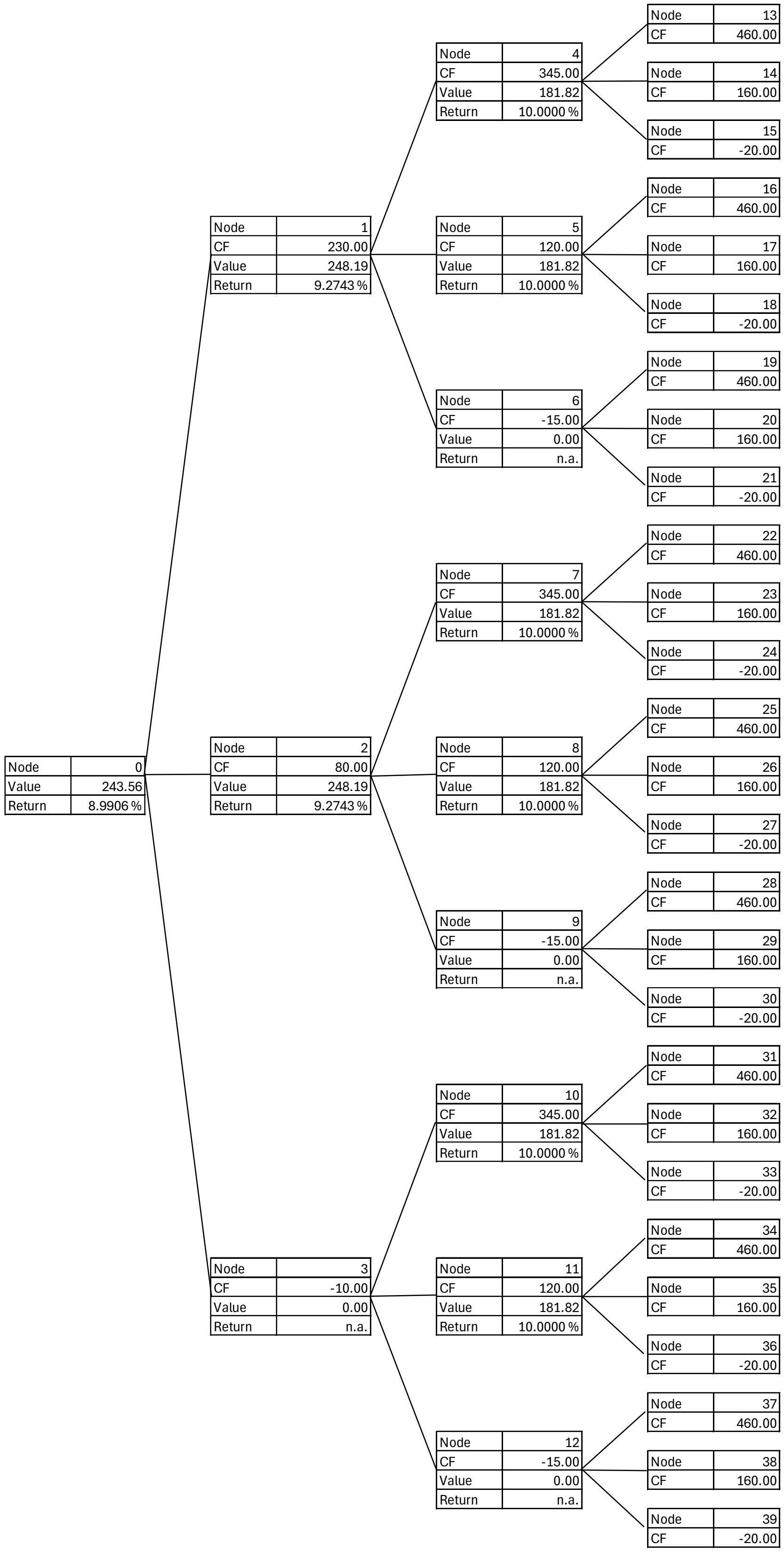

Figure 8: Evolution of cash flows, values, and discount rates for disrupted trend-stationary CF streams

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